

## Error estimates for finite difference-quadrature schemes for a class of nonlocal Bellman equations with variable diffusion

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ABSTRACT. We derive error estimates for certain approximate solutions of Bellman equations associated to a class of controlled jump-diffusion (Lévy) processes. These Bellman equations are fully nonlinear degenerate integro-PDEs interpreted in the sense of viscosity solutions. The approximate solutions are generated by an implicit finite difference-quadrature scheme.

### 1. introduction

We are interested in deriving error estimates for finite difference-quadrature schemes for fully nonlinear degenerate elliptic integro-partial differential equations (integro-PDEs) of Bellman type. These equations are of the form

$$(1.1) \quad H(x, u(x), Du(x), D^2u(x), u(\cdot)) = 0 \text{ in } \mathbb{R}^d.$$

The nonlocal feature of the equation is reflected by the presence of the term  $u(\cdot)$ , where for any  $(x, r, p, X) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$  and for any ‘sufficiently well behaved’  $\varphi$ , the nonlinear functional  $H$ , the Hamiltonian, is defined as

$$H(x, r, p, X, \varphi(\cdot)) = \sup_{\theta \in \Theta} \left\{ \frac{1}{2} \text{tr}[a^\theta(x)X] + b^\theta(x) \cdot p + \mathcal{I}^\theta \varphi - c^\theta(x)r + f^\theta(x) \right\},$$

where the integral operator  $\mathcal{I}^\theta$  is defined as

$$(1.2) \quad \mathcal{I}^\theta(\varphi)(x) = \int_E [\varphi(x + \eta^\theta(z)) - \varphi(x) - \mathbf{1}_{|z| < 1} \eta^\theta(z) \cdot D\varphi(x)] \nu(dz).$$

We denote the set of  $d \times d$  symmetric matrices  $X = (X_{ij}), i, j = 1, 2, \dots, d$  by  $\mathbb{S}$  and  $\Theta$  (the values of the admissible controls) is assumed to be a complete separable metric space. We assume that the  $\mathbb{R}^{d \times d}$ -valued function  $a^\theta(x)$ , the  $\mathbb{R}^d$ -valued functions  $b^\theta(x), \eta^\theta(z)$  and the  $\mathbb{R}$ -valued functions  $c^\theta, f^\theta$  are ‘sufficiently’ regular. In the sequel we will be primarily dealing with a specific form of the

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Hamiltonian  $H$  and the assumptions on the ‘coefficients’ will become clear once this form is revealed. In (1.2),  $\nu(dz)$  is a given Radon measure on

$$E = \mathbb{R}^M \setminus \{0\},$$

the so-called Lévy measure, which typically possesses a second order singularity at the origin and some exponential decay property at infinity. We call (1.1) ‘degenerate elliptic’ since we require the diffusion matrices  $a^\theta(x)$  merely to be nonnegative definite. Additionally, we only require  $|\eta^\theta| \geq 0$ , so  $\eta^\theta(z)$  can be zero for some  $z$ .

In view of this, in general we can not expect the equation (1.1) to have a smooth solution in the classical sense and hence it is necessary to look for a weaker notion of solution. Due to the fully nonlinear and degenerate structure of the problem, a natural choice is to interpret the solution in the viscosity sense. The viscosity solution theory for second order PDEs is now highly developed [4, 5, 14, 15] and there has been a growing interest in the recent years in developing a similar theory for integro-PDEs [1, 2, 3, 6, 11, 10, 12, 17, 18, 24].

Nonlocal equations such as (1.1) arise when one attempts to solve stochastic optimal control problems with the dynamic programming approach [15]. Examples include various types of portfolio optimization problems in which the risky asset follows a jump-diffusion (Lévy) process possessing discontinuous sample paths, see for example [10, 12, 11] and the references therein. The value function of such a control problem is the unique viscosity solution of a Bellman equation of the form (1.1). The regularity and growth properties of the value function depend on the coefficients of the jump-diffusion process [17]. Herein we work with class of problems for which the viscosity solutions are bounded and Lipschitz continuous.

In this paper we focus on finite difference-quadrature schemes for the nonlocal equation (1.1) and their convergence properties. The main idea behind designing such schemes is to replace the original controlled Markov process by a controlled Markov chain (discrete in time). The value function of this discrete-time control problem actually satisfies a dynamic programming principle which results in a finite difference type approximation for the original Bellman equation. In principle, the finite difference equation could be solved by various fixed point iterations.

There is a considerable amount of literature available addressing the issue of convergence of approximate (numerical) solutions to second order PDEs in the viscosity solution framework, see for example [4, 5, 9, 14, 15, 23]. But the question of error estimate for numerical schemes, including finite difference schemes, is much more difficult and remained open until the recent works by Krylov [22, 20, 21] and Barles and Jakobsen [8, 16, 7].

The problem of error estimates for numerical schemes for fully nonlinear degenerate integro-PDEs is mostly an untouched area, except for the recent treatment in [19]. In [19], adapting the methods of Krylov and Barles-Jakobsen, the authors derive error estimates for a class of approximation schemes for (1.1). However, to apply the results from [19] to a particular numerical scheme certain nontrivial conditions have to be satisfied, conditions which could be heard to verify in general. As a working example the authors consider an upwind finite difference-quadrature scheme for the Bellman equation where the diffusion matrix  $a^\theta$  is independent of the spatial variable  $x$  for each admissible control  $\theta$ . It is not clear, from [19], what happens if the diffusion matrix depends on the spatial variable.

In this paper we allow the diffusion matrix to depend on the spatial variable  $x$ , at least for a class of Hamiltonians, and give an error estimate for a finite difference-quadrature scheme which, in some sense, is compatible with the structure of the Hamiltonian  $H$ . Our work extends the results of Krylov [22] to a nonlocal setting. We point out that with methods used herein we are not able to include the case where the jump-vector  $\eta^\theta$  in (1.2) depends explicitly on the spatial variable  $x$ . However, this case as well as parabolic integro-PDEs will be analyzed in [13].

Throughout this paper we make the assumption that the Lévy measure  $\nu(dz)$  sitting inside the integral operator (1.2) is bounded and compactly supported. For the general case where the Lévy measure can be unbounded and have unbounded support, we do not directly discretize the integral operator as it is stated in (1.2). In particular, when the Lévy measure has a second order singularity the origin, we follow [19] and replace (1.1) by an approximate equation in which the singularity has been replaced by an additional diffusion term. To be more precise, for small  $r > 0$  and  $R > 0$  large enough we introduce the truncated domain  $\{z : r < |z| < R\}$  and a truncated Lévy measure  $\nu_{r,R}(dz)$ :

$$\nu_{r,R}(dz) = \mathbf{1}_{r < |z| < R} \nu(dz).$$

The next step would be to replace the measure  $\nu(dz)$  in the integral operator  $\mathcal{I}^\theta$  by  $\nu_{r,R}(dz)$  and modify the diffusion coefficients to account for the singularity at the origin (i.e., the infinite activity region of the Lévy process). We do not explain this aspect in detail here, but refer instead to [19].

Under the assumption that the Lévy measure is bounded and compactly supported, we can alter the form of the Hamiltonian  $H$  without changing the equation (1.1), possibly at the expense of changing  $b^\theta(x)$ . Naming the new Hamiltonian by  $F$  we write

$$F(x, r, p, X, \varphi(\cdot)) = \sup_{\theta \in \Theta} \left\{ \frac{1}{2} \text{tr} [a^\theta(x) X] + b^\theta(x) \cdot p + \mathcal{J}^\theta \varphi - c^\theta(x)r + f^\theta(x) \right\}$$

where the integral operator  $\mathcal{J}^\theta$  is defined as

$$\mathcal{J}^\theta(\varphi)(x) = \int_E [\varphi(x + \eta^\theta(z)) - \varphi(x)] \nu(dz).$$

Then (1.1) takes the form,

$$(1.3) \quad F(x, u(x), Du(x), D^2u(x), u(\cdot)) = 0 \text{ in } \mathbb{R}^d.$$

From now on (1.3) is going to be our focal point.

The remaining part of this paper is organized as follows: In Section 2 we introduce some notation, the notion of viscosity solutions, and define the finite difference-quadrature scheme. In Section 3 we prove existence, uniqueness, and comparison properties for this scheme. Section 4 contains the heart of the matter and here we give the key estimate which leads to continuous dependence and Lipschitz continuity results for the finite difference-quadrature scheme. Finally, in Section 5 we use Krylov's method of shaking the coefficients along with the results of the previous sections to prove the final error estimate.

## 2. Preliminary material and the numerical scheme

We denote a point  $x \in \mathbb{R}^d$  as  $x = (x^1, x^2, \dots, x^d)$ . For any  $l \in \mathbb{R}^d$ , we let  $D_l u$  and  $D_l^2 u$  be the first and second order derivatives of a function  $u$  along the vector

$l$ , i.e.

$$D_l u = u_{x^i} l^i \text{ and } D_l^2 u = u_{x^i x^j} l^i l^j,$$

where  $i$  and  $j$  run from 1 to  $d$  and the standard summation convention applies. The space of bounded Lipschitz continuous functions on  $\mathbb{R}^d$  is denoted by  $C_b^1(\mathbb{R}^d)$  or simply as  $C_b^1$ . For any  $a \in \mathbb{R}$ , we define  $a_{\pm} = a^{\pm} = \frac{1}{2}(|a| \pm a)$ .

For each  $\theta \in \Theta$  and  $k = \pm 1, \pm 2, \dots, \pm d_1$ , where  $d_1$  is a positive integer, we are given vectors  $l_k \in \mathbb{R}^d$ , real-valued functions  $\sigma_k^\theta(x), b_k^\theta(x), f^\theta(x), c^\theta(x)$  on  $\mathbb{R}^d$ , and a  $\mathbb{R}^d$ -valued function  $\eta^\theta(z)$  on  $\mathbb{R}^M$ , satisfying

$$l_k = -l_{-k}, \quad \sigma_k^\theta(x) = \sigma_{-k}^\theta(x).$$

We make the following assumptions:

(A.1) There exist constants  $K > 1$  and  $\lambda > 0$  such that for all  $\theta \in \Theta$

$$\sum_k \{ |l_k| + |\sigma_k^\theta|_{C_b^1} + |b_k^\theta|_{C_b^1} \} + |c^\theta|_{C_b^1} + |f^\theta|_{C_b^1} \leq K,$$

$$c^\theta \geq \lambda, \quad b_k^\theta \geq 0 \quad \text{on } \mathbb{R}^d.$$

(A.2)  $\nu$  is a positive Radon measure on  $E$  satisfying

$$\int_E \nu(dz) < \infty \quad \text{and} \quad \int_{E \setminus B(0, K)} \nu(dz) = 0.$$

(A.3)  $\eta^\theta(z)$  is measurable in  $z$  and satisfies  $\sup\{|\eta^\theta(z)| : 0 < |z| < K, \theta \in \Theta\} \leq K$ , where  $K$  is given in (A.1).

(A.4)  $\sigma_k^\theta, b_k^\theta, c^\theta, f^\theta, \eta^\theta$  are continuous in  $\theta$ .

Next we are going to specify the structure of the diffusion matrix  $a^\theta(x)$  and the drift vector  $b^\theta$  in terms of the functions introduced above:

$$\begin{aligned} \sigma_{ik}^\theta(x) &= l_k^i \sigma_k^\theta(x), & (\sigma^\theta(x)) &= (\sigma_{ik}^\theta)_{i,k} \in \mathbb{R}^{d \times 2d_1}, \\ a^\theta(x) &= \frac{1}{2} \sigma^\theta (\sigma^\theta)^T, & b^\theta(x) &= \sum_r b_r^\theta(x) l_r. \end{aligned}$$

Now the integro-PDE (1.3) can be rewritten in the following form:

$$(2.1) \quad \sup_{\theta \in \Theta} \left\{ \mathcal{L}^\theta u(x) + f^\theta(x) + \mathcal{J}^\theta u(x) \right\} = 0,$$

where  $\mathcal{L}^\theta$  is defined as follows:

$$\mathcal{L}^\theta u(x) = a_k^\theta(x) D_{l_k}^2 u(x) + b_k^\theta(x) D_{l_k} u(x) - c^\theta(x) u(x), \quad a_k^\theta(x) = \frac{1}{2} (\sigma_k^\theta(x))^2.$$

As pointed out in the introduction, in general (2.1) does not possess classical solutions and we need the weaker concept of viscosity solutions.

**DEFINITION 2.1.** A function  $u \in USC(\mathbb{R}^d)$  ( $u \in LSC(\mathbb{R}^d)$ ) is a viscosity subsolution (supersolution) of (2.1) if for every  $x \in \mathbb{R}^d$  and  $\phi \in C^2(\mathbb{R}^d)$  such that  $x$  is a global maximizer (global minimizer) for  $u - \phi$ ,

$$\sup_{\theta \in \Theta} [a_k^\theta D_{l_k}^2 \phi + b_k^\theta D_{l_k} \phi - c^\theta u + \mathcal{J}^\theta \phi + f^\theta(x)] \geq 0 \quad (\leq 0) \quad \text{at } x.$$

We say that  $u$  is a viscosity solution of (2.1) if  $u$  is simultaneously a subsolution and supersolution of (2.1).

As in the case of second order PDEs, it is possible to have a different but equivalent definition in terms of the so-called semijets [14]. Then with the help of the 'maximum principle for semi-continuous functions' [14], suitably adapted to integro-PDEs [17, 18], it is standard to prove existence, uniqueness, and regularity results, even for singular measures  $\nu$  [6, 25, 17, 19, 18]. The results we need for this paper will be stated without proofs.

**THEOREM 2.1.** *Suppose assumptions (A.1), (A.2), and (A.3) hold.*

- (i) *There exists unique viscosity solution  $u \in C_b(\mathbb{R}^d)$  of equation (2.1) which is Hölder continuous, i.e., there are constants  $\delta \in (0, 1]$  and  $C$  such that*

$$|u(x) - u(y)| \leq C|x - y|^\delta \quad \text{for all } x, y \in \mathbb{R}^d.$$

- (ii) *There exists a constant  $\lambda_0$  depending only on  $d, d_1$  and  $K$  such that if*

$$\lambda > \lambda_0,$$

*then the viscosity solution  $u$  of (2.1) is Lipschitz continuous.*

- (iii) *Let  $u, -v \in USC_b(\mathbb{R}^d)$ . If  $u$  and  $v$  are viscosity sub- and supersolutions of (2.1), respectively, then*

$$u \leq v \quad \text{in } \mathbb{R}^d.$$

In this paper we will work with Lipschitz solutions, so in the rest the paper we assume that  $\lambda \geq \lambda_0$ , i.e., we replace assumption (A.1) by

**(A.1')** Assume that assumption (A.1) holds and in addition that

$$c^\theta(x) > \lambda_0,$$

where  $\lambda_0$  is defined in Theorem 2.1.

Next we describe the numerical scheme, which is based on finite differences and numerical quadrature. First, we introduce two finite difference operators. For  $h_1, h_2 > 0, l \in \mathbb{R}^d, x \in \mathbb{R}^d$ ,

$$\begin{aligned} \delta_{h_1, l} u(x) &= \frac{u(x + h_1 l) - u(x)}{h_1}, \\ \Delta_{h_1, l} u(x) &= \frac{u(x + h_1 l) - 2u(x) + u(x - h_1 l)}{h_1^2}. \end{aligned}$$

To discretize the integral operator in (2.1) we introduce a quadrature rule:

$$I_{h_2}(f) = \sum_{p \in h_2 \mathbb{Z}^M} f(p) k_p, \quad k_p \geq 0, \quad k_p = 0 \quad \text{for } |p| > K,$$

where  $p \in h_2 \mathbb{Z}^M$  and  $k_p \geq 0$  are the nodes and weights respectively. Since  $k_p \geq 0$  this scheme is monotone. This assumption is quite natural since the measure  $\nu$  is positive; Of course, it is also needed for the analysis. Note that the sum is a finite sum since  $k_p = 0$  for  $|p| > K$ , and this is natural since the measure  $\nu$  has support in  $|p| < K$ . We also require that the error satisfies

$$(2.2) \quad \left| \int_E f(z) \nu(dz) - I_{h_2}(f) \right| \leq \nu(E) L_f h_2,$$

for every Lipschitz function  $f$  with Lipschitz constant  $L_f$ .

Many classical quadrature rules are of this form, the simplest example being the Riemann sum approximation

$$I_{h_2}(f) = \sum_{p \in h_2 \mathbb{Z}^M} f(p) \nu(p + [0, h_2]^M).$$

Other examples include the Newton-Cotes quadratures of order less than 9, see [19].

Now we are in a position to finally write down the numerical scheme for (2.1). This implicit finite difference-quadrature scheme is given by

$$(2.3) \quad \sup_{\theta \in \Theta} [\mathcal{L}_{h_1}^\theta u + f^\theta(x) + \mathcal{J}_{h_2}^\theta u] = 0 \text{ in } \mathbb{R}^d,$$

where

$$\begin{aligned} \mathcal{L}_{h_1}^\theta u &= a_k^\theta \Delta_{h_1, l_k} u + b_k^\theta \delta_{h_1, l_k} u - c^\theta u \\ \mathcal{J}_{h_2}^\theta u &= I_{h_2}(u(x + \eta^\theta(x, z)) - u(x)). \end{aligned}$$

The scheme (2.1) is compatible with the structure of (2.1) in the sense that, for a given four times continuously differentiable function  $g$  and for each  $\theta \in \Theta$ ,

$$(2.4) \quad |\mathcal{L}_{h_1}^\theta g - \mathcal{L}^\theta g|(x) \leq N^* (h_1^2 \sup_{B_K(x)} |D_x^4 g| + h_1 \sup_{B_K(x)} |D_x^2 g|),$$

where  $N^*$  is constant which only depends on  $K, d_1$ , and  $B_K(x)$  denotes the closed ball of radius  $K$  centered at  $x$ . This relation is a simple consequence of Taylor's theorem.

### 3. Some basic properties of the scheme

In this section we prove the existence and uniqueness of solutions to the finite difference-quadrature scheme (2.3).

LEMMA 3.1. *If assumptions (A.1), (A.2), (A.3) are satisfied, then there exists a unique  $u \in C_b(\mathbb{R}^d)$  solving (2.3).*

PROOF. For a constant  $\epsilon > 0$ , define the nonlinear operator  $G^\epsilon$  on  $C_b(\mathbb{R}^d)$  as follows: For  $u \in C_b(\mathbb{R}^d)$ ,

$$\begin{aligned} G^\epsilon[u](x) := & u(x) + \epsilon \sup_{\theta \in \Theta} \left[ a_k^\theta(x) \Delta_{h_1, l_k} u(x) + b_k^\theta(x) \delta_{h_1, l_k} u(x) - c^\theta(x) u(x) \right. \\ & \left. + f^\theta(x) + \sum_{p \in h_2 \mathbb{Z}^M} k_p (u(x + \eta^\theta(p)) - u(x)) \right]. \end{aligned}$$

Note that  $u \in C_b$  solves (2.3) if and only if it is a fixed point for  $G^\epsilon$  for some positive  $\epsilon$ . We write the expression for  $G^\epsilon$  in the following way,

$$G^\epsilon[u](x) = \sup_{\theta \in \Theta} \left[ w_k^\theta(x) u(x + h_1 l_k) + w^\theta(x) u(x) + \epsilon f^\theta(x) + \sum_{p \in h_2 \mathbb{Z}^M} \epsilon k_p u(x + \eta^\theta(p)) \right],$$

where  $w_k^\theta(x) = 2\epsilon h_1^{-2} a_k^\theta \geq 0$  and

$$w^\theta(x) = 1 - \sum_k w_k^\theta - \epsilon c^\theta - \epsilon \sum_{p \in h_2 \mathbb{Z}^M} k_p.$$

Choose  $\epsilon > 0$  small enough so that  $w^\theta \geq 0$  and  $1 - \epsilon \lambda < 1$ . Observe that

$$0 \leq \sum_k w_k^\theta + w^\theta + \epsilon \sum_{p \in h_2 \mathbb{Z}^M} k_p = 1 - \epsilon c^\theta \leq 1 - \epsilon \lambda.$$

So using the property that a difference of sup's is less than the sup of the difference, for any two  $u, v \in C_b(\mathbb{R}^d)$  we obtain

$$|G^\epsilon[u](x) - G^\epsilon[v](x)| \leq (1 - \epsilon\lambda) \sup_{\mathbb{R}^d} |u - v|.$$

Hence  $G^\epsilon$  is a contraction on the Banach space  $C_b(\mathbb{R}^d)$  equipped with supremum norm, and by applying Banach fixed point theorem we conclude that  $G^\epsilon$  has a unique fixed point  $u \in C_b(\mathbb{R}^d)$ .  $\square$

For a given set of discretization parameters  $h_1, h_2 > 0$ , we denote the vector  $(h_1, h_2)$  by  $h$  and the unique solution of the finite difference-quadrature scheme (2.3) by  $v_h$ . Next, we are going to prove the comparison principle for the scheme.

LEMMA 3.2. *Let  $u, v \in C_b(\mathbb{R}^d)$  such that*

$$(3.1) \quad \sup_{\theta \in \Theta} [\mathcal{L}_{h_1}^\theta u + f^\theta(x) + \mathcal{J}_{h_2}^\theta u] \leq \sup_{\theta \in \Theta} [\mathcal{L}_{h_1}^\theta v + f^\theta(x) + \mathcal{J}_{h_2}^\theta v]$$

for all  $x \in \mathbb{R}^d$ . Then

$$u \geq v \text{ in } \mathbb{R}^d.$$

PROOF. Let

$$m = \sup_{x \in \mathbb{R}^d} (v - u).$$

We want to show that  $m \leq 0$ . To this end, we argue by contradiction and assume  $m > 0$ . Let  $(x_n, \delta_n)$  be a sequence of points in  $\mathbb{R}^{d+1}$  such that  $\delta_n = v(x_n) - u(x_n)$  and  $\delta_n \rightarrow m$  as  $n \rightarrow \infty$ .

Let  $w_k^\theta$  and  $\bar{w}^\theta$  be defined as in the proof of Lemma 3.1. Set  $\varepsilon = 1$  and define  $\bar{w}_k^\theta := w_k^\theta|_{\varepsilon=1}$  and  $\bar{w}^\theta := \bar{w}_k^\theta|_{\varepsilon=1} - 1$ . Then we have

$$\begin{aligned} & \sup_{\theta \in \Theta} [\mathcal{L}_{h_1}^\theta v(x_n) + f^\theta(x_n) + \mathcal{J}_{h_2}^\theta v(x_n)] \\ &= \sup_{\theta \in \Theta} \left[ \sum_k \bar{w}_k^\theta(x_n) v(x_n + h_1 l_k) + \bar{w}^\theta(x_n) v(x_n) + \sum_p k_p v(x_n + \eta(p)) + f^\theta(x_n) \right] \\ &\leq \sup_{\theta \in \Theta} \left[ \sum_k \bar{w}_k^\theta(x_n) (u(x_n + h_1 l_k) + m) + \bar{w}^\theta(x_n) (u(x_n) + \delta_n) \right. \\ &\quad \left. + \sum_p k_p (u(x_n + \eta(p)) + m) + f^\theta(x_n) \right] \\ &= \sup_{\theta \in \Theta} \left[ \mathcal{L}_{h_1}^\theta u(x_n) + f^\theta(x_n) + \mathcal{J}_{h_2}^\theta u(x_n) + \left( \sum_p k_p + \sum_k \bar{w}_k^\theta \right) (m - \delta_n) - c^\theta \delta_n \right] \\ &\leq \sup_{\theta \in \Theta} \left[ \mathcal{L}_{h_1}^\theta u(x_n) + f^\theta(x_n) + \mathcal{J}_{h_2}^\theta u(x_n) \right] + N(h_1, h_2, d_1, K, \sum_p k_p) (m - \delta_n) - \bar{\lambda} \delta_n. \end{aligned}$$

Sending  $n \rightarrow \infty$  and using (3.1) we get

$$\bar{\lambda} m \leq 0$$

which contradicts  $m > 0$ .  $\square$

Since  $\pm \frac{1}{\lambda} \sup_{\theta \in \Theta} |f^\theta|_{C_b}$  are super/sub solutions of (2.3) we get immediately

COROLLARY 3.3. *If  $u_h \in C_b(\mathbb{R}^d)$  is the unique solution of (2.3), then*

$$|u_h|_{C_b} \leq \frac{1}{\lambda} \sup_{\theta \in \Theta} |f^\theta|_{C_b}.$$

This bound does not depend on  $h$ .

#### 4. Lipschitz continuity and continuous dependence estimates for the finite difference-quadrature scheme

The following theorem is the most crucial estimate of this paper.

**THEOREM 4.1.** *Assume (A.1), (A.2), (A.3), (2.2) hold and let  $v(x)$  be the unique solution of (2.3). Then there is a constant  $\bar{\lambda}_0 \in (0, \infty)$ , depending only on  $K, d_1, d$ , and the Levy measure  $\nu$ , such that if*

$$\lambda > \bar{\lambda}_0,$$

then there is a constant  $N$ , depending on the data, such that

$$|\delta_{\epsilon, \pm l} v| \leq N,$$

for every  $0 < \epsilon < h_1$  and  $l \in \mathbb{R}^d$  with  $|l| \leq K$ .

**PROOF.** We start by introducing the notation  $h_k = h_1$  for  $k = \pm 1, \pm 2, \dots, \pm d_1$ ,  $h_{\pm(d_1+1)} = \epsilon$ ,  $l_{\pm(d_1+1)} = \pm l$ . Moreover, we let  $r$  be an index running through  $\{\pm 1, \pm 2, \dots, \pm(d_1 + 1)\}$  and  $k$  be an index running through  $\{\pm 1, \pm 2, \dots, \pm d_1\}$ . Finally, we let

$$v_r = \delta_{h_r, l_r} v, \quad v_r^- = (v_r)_-,$$

and

$$W(x) = \sum_r (v_r^-)^2.$$

As it is argued for in Krylov [22], to prove the theorem it is enough to show that the newly introduced quantity  $W(x)$  is bounded. Note that trivially  $|W(x)| \leq \frac{1}{\min_r (h_r)} N |v|_{C_b}$ . We want to prove that for a big enough  $\lambda$ ,  $|W(x)|$  can be bounded independently of the parameters  $h_1, h_2, \epsilon$  etc. To do this we introduce

$$V(x) = W(x) - \delta C(x)$$

where  $C(x) \in C^2(\mathbb{R}^d)$  is positive and satisfies

$$\lim_{|x| \rightarrow \infty} C(x) = \infty \quad \text{and} \quad |\nabla C|_{C_b} + |\nabla^2 C|_{C_b} \leq K.$$

It is clear that  $V(x)$  is bounded above and that there exists a point  $x_0 \in \mathbb{R}^d$  such that  $V(x)$  attains its maximum at  $x_0$ .

From equation (2.3) it follows that there is a sequence  $(\theta_n) \subset \Theta$  such that, at  $x_0$ ,

$$\lim_{n \rightarrow \infty} \left[ a_k^{\theta_n} \Delta_{h_1, l_k} v + b_k^{\theta_n} \delta_{h_1, l_k} v - c^{\theta_n} v + f^{\theta_n} + \sum_p k_p (v(x_0 + \eta^{\theta_n}(p)) - v(x_0)) \right] = 0.$$

Now, by assumptions (A.1), (A.2), (A.3) and Arzela-Ascoli's theorem, there exists a subsequence of  $(\theta_n)$ , which we do not bother to relabel, and functions  $\bar{a}_k, \bar{b}_k, \bar{c}, \bar{f}, \bar{\eta}$ , satisfying assumptions (A.1), (A.2), (A.3), and

$$\left( \bar{a}_k, \bar{b}_k, \bar{c}, \bar{f}, \bar{\eta} \right) = \lim_{n \rightarrow \infty} \left( a_k^{\theta_n}, b_k^{\theta_n}, c^{\theta_n}, f^{\theta_n}, \eta^{\theta_n} \right).$$

Obviously, at the point  $x_0$ ,

$$(4.1) \quad \bar{a}_k \Delta_{h_1, l_k} v + \bar{b}_k \delta_{h_1, l_k} v - \bar{c} v + \bar{f} + \sum_p k_p (v(x_0 + \bar{\eta}(p)) - v(x_0)) = 0,$$

whereas at points of the form  $(x_0 + h_r l_r)$  (and every other point) we have the inequality

$$(4.2) \quad \bar{a}_k \Delta_{h_1, l_k} v + \bar{b}_k \delta_{h_1, l_k} v - \bar{c} v + \bar{f} + \sum_p k_p (v(x_0 + h_r l_r + \bar{\eta}(p)) - v(x_0 + h_r l_r)) \leq 0.$$

Subtracting (4.1) from (4.2) and dividing throughout by  $h_r$ , we get

$$(4.3) \quad \bar{a}_k \Delta_{h_k, l_k} v_r + I_{1r} + I_{2r} + I_{3r} + I_{4r} + I_{5r} \leq 0,$$

where

$$\begin{aligned} I_{1r} &= (\delta_{h_r, l_r} \bar{a}_k) \Delta_{h_k, l_k} v, \\ I_{2r} &= h_r (\delta_{h_r, l_r} \bar{a}_k) \Delta_{h_k, l_k} v_r, \\ I_{3r} &= (T_{h_r, l_r} \bar{b}_k) \delta_{h_k, l_k} v_r + (\delta_{h_r, l_r} \bar{b}_k) \delta_{h_k, l_k} v, \\ I_{4r} &= -(\delta_{h_r, l_r} \bar{c}) v - (T_{h_r, l_r} \bar{c}) v_r + \delta_{h_r, l_r} \bar{f}, \\ I_{5r} &= \sum_{p \in h_2 \mathbb{Z}^M} k_p (v_r(x_0 + \bar{\eta}(p)) - v_r). \end{aligned}$$

Next, we multiply (4.3) by  $v_r^-$  and sum the result with respect to  $r$ . The main step of the proof is to estimate the different terms in (4.3) as they appear after multiplication with  $v_r^-$  and summation with respect to  $r$ . Except for the  $I_{5r}$ -term, these terms can be estimated as in Krylov [22] with obvious modifications (see [13]). The result is that at  $x_0$ ,

$$(4.4) \quad \sum_r v_r^- I_{4r} \geq -N(d_1, d, K) + (\lambda - 2K)W,$$

$$(4.5) \quad \sum_r v_r^- [\bar{a}_k \Delta_{h_k, l_k} v_r + I_{1r} + I_{2r} + I_{3r}] \geq -N(d, d_1, K)(V + 1) - \mathcal{O}(\delta).$$

Now we estimate the “new” term  $v_r^- I_{5r}$ :

$$\sum_r v_r^- I_{5r} = \left( \sum_p k_p \right) \left( \sum_r (v_r^-)^2 \right) + \sum_p \sum_r k_p v_r^- v_r(x_0 + \bar{\eta}(p)).$$

By assumptions (A.2) and (2.2) (implying  $\nu(E) = (\sum_p k_p)$ ),

$$\begin{aligned} \sum_r v_r^- I_{5r} &= \nu(E)W + \sum_p \sum_r k_p v_r^- v_r(x_0 + \bar{\eta}(p)) \\ &\geq \nu(E)W - \frac{1}{2} \sum_p \sum_r k_p [(v_r^-)^2 + v_r^2(x_0 + \bar{\eta}(p))] \\ (4.6) \quad &= \frac{\nu(E)}{2}W - \frac{1}{2} \sum_p \left( k_p \sum_r v_r^2(x_0 + \bar{\eta}(p)) \right). \end{aligned}$$

The next step is to estimate

$$\sum_r v_r^2(x_0 + \bar{\eta}(p))$$

for each  $p$ . Note that, if  $v_r(x_0 + \bar{\eta}(p)) \leq 0$  then,

$$|v_r(x_0 + \bar{\eta}(p))| = v_r^-(x_0 + \bar{\eta}(p)).$$

If  $v_r(x_0 + \bar{\eta}(p)) > 0$ , then

$$0 > -v_r(x_0 + \bar{\eta}(p)) = v_{-r}(x_0 + h_r l_r + \bar{\eta}(p)).$$

Taking both possible situations into account we get

$$v_r^2(x_0 + \bar{\eta}(p)) \leq \left[ (v_r^-)^2(x_0 + \bar{\eta}(p)) + (v_{-r}^-)^2(x_0 + h_r l_r + \bar{\eta}(p)) \right].$$

Summing with respect to  $r$  yields

$$\begin{aligned} & \sum_r v_r^2(x_0 + \bar{\eta}(p)) \\ & \leq \left[ \sum_r (v_r^-)^2(x_0 + \bar{\eta}(p)) \right] + \sum_r (v_{-r}^-)^2(x_0 + h_r l_r + \bar{\eta}(p)) \\ & = W(x_0 + \bar{\eta}(p)) + \sum_r (v_{-r}^-)^2(x_0 + h_r l_r + \bar{\eta}(p)) \\ & \leq W(x_0 + \bar{\eta}(p)) + \sum_r W(x_0 + h_r l_r + \bar{\eta}(p)) \\ & \leq V(x_0) + \delta C(x_0 + \bar{\eta}(p)) + \sum_r [V(x_0) + \delta C(x_0 + h_r l_r + \bar{\eta}(p))] \\ (4.7) \quad & = (2d_1 + 3)V(x_0) + \delta C(x_0 + \bar{\eta}(p)) + \delta \sum_r C(x_0 + h_r l_r + \bar{\eta}(p)). \end{aligned}$$

Combining (4.6) and (4.7) we conclude,

$$\begin{aligned} \sum_r v_r^- I_{5r} & \geq - (d_1 + 1)\nu(E)V(x_0) - \frac{\delta}{2} \sum_p k_p C(x_0 + \bar{\eta}(p)) \\ (4.8) \quad & - \frac{\delta}{2} \sum_p \sum_r k_p C(x_0 + h_r l_r + \bar{\eta}(p)). \end{aligned}$$

Combining (4.3), (4.4), (4.5), and (4.8), we get

$$\lambda V \leq N(d, d_1, K, \nu(E))(V + 1) + \mathcal{O}(\delta).$$

Taking  $\lambda \geq N(d, d_1, K, \nu(E)) + 1$  and sending  $\delta \downarrow 0$ , we get

$$W \leq N(d_1, d, K, \nu(E)),$$

and the theorem is proved.  $\square$

As an application of the above theorem we get the following important result, namely the continuous dependence estimate for the finite difference-quadrature equation (2.3):

**THEOREM 4.2.** *Let  $(\sigma_k^\theta, b_k^\theta, c^\theta, f^\theta, \eta^\theta, \lambda)$  and  $(\hat{\sigma}_k^\theta, \hat{b}_k^\theta, \hat{c}^\theta, \hat{f}^\theta, \hat{\eta}^\theta, \hat{\lambda})$  be two sets of coefficients satisfying assumptions **(A.1)**, **(A.2)**, **(A.3)**, with  $\hat{\lambda} = \lambda > \bar{\lambda}_0$  (defined in Theorem 4.1) and  $\eta^\theta = \hat{\eta}^\theta$ . Assume (2.2) holds, and let  $u$  and  $\hat{u}$  be the solutions of (2.3) with coefficients  $(\sigma_k^\theta, b_k^\theta, c^\theta, f^\theta, \eta^\theta, \lambda)$  and  $(\hat{\sigma}_k^\theta, \hat{b}_k^\theta, \hat{c}^\theta, \hat{f}^\theta, \hat{\eta}^\theta, \hat{\lambda})$  respectively. Introduce the quantity*

$$\epsilon := \sup_{\mathbb{R}^d, A, k} \left\{ |\hat{\sigma}_k^\theta - \sigma_k^\theta| + |\hat{b}_k^\theta - b_k^\theta| + |\hat{c}^\theta - c^\theta| + |\hat{f}^\theta - f^\theta| \right\}.$$

*Then there exists a constant  $N$ , depending only on  $K, d, d_1$ , and  $\nu(E)$ , such that*

$$(4.9) \quad |u - \hat{u}| \leq N\epsilon \quad \text{in } \mathbb{R}^d.$$

We do not prove this theorem here since the proof is similar to the one given by Krylov in [22], see also [13] for a proof in the case of parabolic integro-PDEs.

### 5. Shaking of the coefficients and the final error estimate

In this section we use the method of shaking the coefficients, introduced by Krylov [20, 21], to prove the desired error bound on  $|v - v_h|$ , where the functions  $v$  and  $v_h$  are the solutions to (1.3) and (2.3) respectively.

Let  $S \subset B_1 = \{x \in \mathbb{R}^d : |x| \leq 1\}$ . For  $\epsilon > 0$ , let  $v_h^{\epsilon, S}$  be the unique solution of

$$(5.1) \quad \sup_{(\theta, y) \in \Theta \times S} \left[ \mathcal{L}_{h_1}^\theta(x + \epsilon y)u(x) + f^\theta(x + \epsilon y) + \mathcal{J}_{h_2}^\theta u \right] = 0$$

and  $v^{\epsilon, S}$  be the unique viscosity solution of

$$\sup_{(\theta, y) \in \Theta \times S} \left[ \mathcal{L}^\theta(x + \epsilon y)u(x) + f^\theta(x + \epsilon y) + \mathcal{J}^\theta u \right] = 0.$$

We have the following lemma:

LEMMA 5.1. *There is a constant  $N$ , depending only on  $d, d_1, K, \nu(E)$ , such that if assumptions (A.1), (A.2), (A.3), (2.2) are satisfied with  $\lambda > \max(\lambda_0, \bar{\lambda}_0)$  ( $\lambda_0, \bar{\lambda}_0$  are defined in Theorems 2.1 and 4.1 respectively), then*

$$(5.2) \quad |v_h^{\epsilon, S} - v_h| \leq N\epsilon,$$

$$(5.3) \quad |v^{\epsilon, S} - v| \leq N\epsilon.$$

In particular, upon choosing  $S = \frac{y-x}{|y-x|}$  and  $\epsilon = |x - y|$ , we have

$$\begin{aligned} |v_h(x) - v_h(y)| &\leq N|x - y|, \\ |v_h^{\epsilon, S}(x) - v_h^{\epsilon, S}(y)| &\leq N|x - y|. \end{aligned}$$

PROOF. Estimate (5.2) is a trivial consequence of (4.9) and for a proof of (5.3) we refer to the appendix of [19].  $\square$

THEOREM 5.2 (Error estimate). *If assumptions (A.1), (A.2), (A.3), (2.2), are satisfied with  $\lambda > \max(\lambda_0, \bar{\lambda}_0)$  ( $\lambda_0$  and  $\bar{\lambda}_0$  are defined in Theorems 2.1 and 4.1 respectively), then there is a constant  $N$ , depending on the data but not  $h$ , such that*

$$(5.4) \quad |v - v_h| \leq N(h_1^{\frac{1}{2}} + h_2).$$

PROOF. Since  $v, v_h$  are bounded and the bound is independent of  $(h_1, h_2)$ , without loss of generality we can assume that  $h_1, h_2 \leq 1$ . Let  $(\rho_\epsilon)_{\epsilon > 0}$  be a family of standard mollifiers on  $\mathbb{R}^d$ . Upon choosing  $S = B_1$ , we denote  $v_h^{\epsilon, S}$  by  $v_h^\epsilon$ ,  $v^{\epsilon, S}$  by  $v^\epsilon$ , and define  $v_\epsilon = \rho_\epsilon * v^\epsilon$ . Using the same techniques as in the proof of Theorem 7.4 of [19], we find  $v_\epsilon$  to be a viscosity supersolution to (2.1), i.e., for each  $\theta \in \Theta$

$$\mathcal{L}^\theta(x)v_\epsilon + f^\theta(x) + \mathcal{J}^\theta v_\epsilon \leq 0.$$

Then from (2.4), (2.2), and properties of mollifiers we get

$$\mathcal{L}_{h_1}^\theta(x)v_\epsilon + f^\theta + \mathcal{J}_{h_2}^\theta v_\epsilon \leq N\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2\right)$$

for all  $\theta \in \Theta$ . So  $v_\epsilon + \frac{N}{\lambda}\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2\right)$  is a supersolution of the scheme (2.3), and the comparison principle (Lemma 3.2) yields

$$v_h \leq v_\epsilon + \frac{N}{\lambda_0}\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2\right) \leq v + N\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2 + \epsilon\right).$$

Now choose  $\epsilon = h_1^{\frac{1}{2}}$  in the above inequality to get  $v_h - v \leq N(h_1^{\frac{1}{2}} + h_2)$ .

Next we want to get a similar bound on  $v - v_h$ . After a change of variable in (5.1) we get, for each  $\theta$ ,

$$(5.5) \quad \mathcal{L}_{h_1}^\theta(x)v_h^\epsilon(x - \epsilon y) + f^\theta(x) + \mathcal{J}_{h_2}^\theta v_h^\epsilon(x - \epsilon y) \leq 0.$$

Multiplying (5.5) by  $\rho_\epsilon(y)$  and integrating with respect  $y$  we get

$$\mathcal{L}_{h_1}^\theta(x)(v_h^\epsilon * \rho_\epsilon) + f^\theta(x) + \mathcal{J}_{h_2}^\theta(v_h^\epsilon * \rho_\epsilon) \leq 0.$$

Once again by properties of mollifiers along with Lemma 5.1 and (2.2), (2.4) we obtain

$$\mathcal{L}^\theta(x)(v_h^\epsilon * \rho_\epsilon) + f^\theta(x) + \mathcal{J}^\theta(v_h^\epsilon * \rho_\epsilon) \leq N\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2\right)$$

for all  $\theta \in \Theta$ . So  $v_h^\epsilon * \rho_\epsilon + \frac{N}{\lambda_0}\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2\right)$  is a supersolution of equation (2.1) and by the comparison principle (Theorem 2.1),

$$v \leq v_h^\epsilon * \rho_\epsilon + \frac{N}{\lambda_0}\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2\right).$$

By Lemma 5.1 and properties of mollifiers, we get

$$v \leq v_h^\epsilon * \rho_\epsilon + N\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2\right) \leq v_h + N\left(\frac{h_1^2}{\epsilon^3} + \frac{h_1}{\epsilon} + h_2 + \epsilon\right).$$

Once again we replace  $\epsilon$  by  $h_1^{\frac{1}{2}}$  which yields  $v - v_h \leq N(h_1^{\frac{1}{2}} + h_2)$ .  $\square$

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