ON NUMERICAL METHODS AND ERROR ESTIMATES FOR DEGENERATE FRACTIONAL CONVECTION-DIFFUSION EQUATIONS

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Abstract. First we introduce and analyze a convergent numerical method for a large class of nonlinear nonlocal possibly degenerate convection diffusion equations. Secondly we develop a new Kuznetsov type theory and obtain general and possibly optimal error estimates for our numerical methods – even when the principal derivatives have any fractional order between 1 and 2! The class of equations we consider includes equations with nonlinear and possibly degenerate fractional or general Levy diffusion. Special cases are conservation laws, fractional conservation laws, certain fractional porous medium equations, and new strongly degenerate equations.

1. Introduction

In this paper we develop a numerical method along with a general Kuznetsov type theory of error estimates for integro partial differential equations of the form

$$\begin{cases} \partial_t u + \mathrm{div} f(u) = \mathcal{L}^\mu[A(u)], & (x,t) \in Q_T, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $Q_T = \mathbb{R}^d \times (0,T)$ and the nonlocal diffusion operator $\mathcal{L}^\mu$ is defined as

$$\mathcal{L}^\mu[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot \nabla\phi(x)\, \mathbf{1}_{|z|<1}(z)\ \mathrm{d}\mu(z), \tag{1.2}$$

for smooth bounded functions ϕ. Here $\mathbf{1}$ denotes the indicator function. Throughout the paper the data (f, A, μ, u_0) is assumed to satisfy:

($\mathbf{A}$.1) $f = (f_1, \dots, f_d) \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^d)$ with $f(0) = 0$,

($\mathbf{A}$.2) $A \in W^{1,\infty}(\mathbb{R})$, A non-decreasing with $A(0) = 0$,

($\mathbf{A}$.3) $\mu \geq 0$ is a Radon measure such that $\int_{|z|>0} |z|^2 \wedge 1\ \mathrm{d}\mu(z) < \infty$,

($\mathbf{A}$.4) $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$.

We use the notation $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$.

Remark 1.1. These assumptions can be relaxed in two standard ways: (i) f, A can take any value at $u = 0$ (replace f by $f - f(0)$ etc.), and (ii) f, A can be assumed to be locally Lipschitz. By the maximum principle and ($\mathbf{A}$.4), solutions of (1.1) are bounded, and locally Lipschitz functions are Lipschitz on compact domains.

The measure μ and the operator $\mathcal{L}^\mu$ are respectively the Lévy measure and the generator of a pure jump Lévy process. Any such process has a Lévy measure

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and generator satisfying (1.2) and (**A**.3), see e.g. [4]. Example are the symmetric α-stable processes with fractional Laplace generators where

$$d\mu(z) = c_\lambda \frac{dz}{|z|^{d+\lambda}} \quad (c_\lambda > 0) \qquad \text{and} \qquad \mathcal{L}^\mu \equiv -(-\Delta)^{\lambda/2} \quad \text{for } \lambda \in (0,2). \tag{1.3}$$

Non-symmetric examples are popular in mathematical finance, e.g. the CGMY model where

$$d\mu(z) = \begin{cases} \dfrac{C\, e^{-G|z|}}{|z|^{1+\lambda}} dz & \text{for } z > 0, \\ \dfrac{C\, e^{-M|z|}}{|z|^{1+\lambda}} dz & \text{for } z < 0, \end{cases}$$

and where $d = 1$, $\lambda (= Y) \in (0, 2)$, and $C, G, M > 0$. We refer the reader to [14] for more details on this and other nonlocal models in finance. In both examples the nonlocal operator behaves like a fractional derivative of order between 0 and 2.

Equation (1.1) has a local non-linear convection term (the f-term) and a fractional (or nonlocal) non-linear possibly degenerate diffusion term (the A-term). Special cases are scalar conservation laws ($A \equiv 0$), fractional and Lévy conservation laws ($A(u) = u$ and α-stable or more general μ) – see e.g. [6, 1] and [7, 29, 25], fractional porous medium equations [16] ($A = |u|^{m-1}u$ for $m \geq 1$ and α-stable μ), and strongly degenerate equations where A vanishes on a set of positive measure. If either A is degenerate or $\mathcal{L}^\mu$ is a fractional derivative of order less than 1, then solutions of (1.1) are not smooth in general and uniqueness fails for weak (distributional) solutions. Uniqueness can be regained by imposing additional entropy conditions in a similar way to what is done for conservation laws. The Kruzkov entropy solution theory of scalar conservation laws [27] was extended to cover fractional conservation laws in [1], to more general Lévy conservation laws in [25], and then finally to setting of this paper, equations with non-linear fractional diffusion and general Lévy measures in [11]. For local 2nd order degenerate convection diffusion equations like

$$\partial_t u + \text{div} f(u) = \Delta A(u), \tag{1.4}$$

there is an entropy solution theory due to Carrillo [9].

In recent years, integro partial differential equations like (1.1) have been at the center of a very active field of research. A thorough description of the mathematical background for such equations, relevant bibliography, and applications to several disciplines of interest can be found in [1, 2, 7, 11, 16, 25].

The first contribution of this paper is to introduce a numerical method for equation (1.1) and prove that it converges toward the entropy solution of (1.1) under assumptions (**A**.1)–(**A**.4). The numerical method is based upon a monotone finite volume discretization of an approximate equation with truncated and hence bounded Lévy measure. Essentially it is an extension of the method in [11] from symmetric α-stable to general Lévy measures, but since non-symmetric measures are allowed, the discretization becomes more complicated here. Apart from its ability to capture the correct solution for the whole family of equations of the form (1.1), the main advantage of our numerical method is that it allows for a complete error analysis through the new framework for error estimates that we develop in the second part of the paper.

The second, and probably most important contribution of the paper, is the development of a theory capable of producing error estimates for degenerate equations of order greater than 1. This theory is based on a non-trivial extension of the Kuznetsov theory for scalar conservation laws [28] to the current fractional diffusion setting. An initial step in this analysis was performed in [2], with the derivation of a so-called Kuznetsov lemma in a relevant form for (1.1). In [2] the lemma is used in the derivation of continuous dependence estimates and error estimates for vanishing viscosity type of approximations of (1.1). In the present paper, we show how it can be used in solving the more difficult problem of finding error estimates for numerical methods for (1.1).

As a corollary of our Kuznetsov type theory, we obtain explicit λ-dependent error estimates when μ is a measure satisfying

$$0 \leq \mathbf{1}_{|z|<1} d\mu(z) \leq c_\lambda \frac{dz}{|z|^{d+\lambda}} \qquad \text{for} \qquad c_\lambda > 0 \text{ and } \lambda \in (0,2). \tag{1.5}$$

In this paper we will call such measures *fractional measures*. For example for the implicit version of our numerical method (3.5), we prove in Section 6 that

$$\|u(\cdot,T) - u_{\Delta x}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq C_T \begin{cases} \Delta x^{\frac{1}{2}} & \lambda \in (0,1), \\ \Delta x^{\frac{1}{2}} \log(\Delta x) & \lambda = 1, \\ \Delta x^{\frac{2-\lambda}{2}} & \lambda \in (1,2), \end{cases}$$

where u is the entropy solution of (1.1) and $u_{\Delta x}$ is the solution of (3.5). Note that our error estimate covers all values $\lambda \in (0,2)$, all spacial dimensions d, and possibly strongly degenerate equations! Also note that under our assumptions, the solution u possibly only have BV regularity in space. Hence the error estimate is robust in the sense that it holds also for discontinuous solutions, and moreover, the classical result of Kuznetsov [28] for conservation laws follows as a corollary by taking $A \equiv 0$ (a valid choice here!) and $\lambda \in (0,1)$. The above estimate is also consistent with error estimates for the vanishing λ-fractional viscosity method,

$$\partial_t u + \mathrm{div} f(u) = -\Delta x\, (-\Delta)^{\lambda/2} u \quad \text{as} \quad \Delta x \to 0^+,$$

see e.g. [18, 1], but note that our problem is different and much more difficult.

There is a vast literature on approximation schemes and error estimates for scalar conservations laws, we refer e.g. to the books [26, 22] and references therein for more details. For local degenerate convection-diffusion equations like (1.4), some approximation methods and error estimates can be found e.g. in [20, 21, 24] and references therein. In this setting it is very difficult to obtain error estimates for numerical methods, and the only result we are aware of is a very recent one by Karlsen et al. [24] (but see also [10]). This very nice result applies to rather general equations of the form (1.4) but in one space dimension and under additional regularity assumptions (e.g. $\partial_x(A(u)) \in BV$). When it comes to nonlocal convection-diffusion equations, the literature is very recent and not yet very extensive. The paper [15] introduce finite volume schemes for radiation hydrodynamics equations, a model where $\mathcal{L}^\mu$ is a nonlocal derivative of order 0. Then fractional conservation laws are discretized in [17, 13, 12] with finite difference, discontinuous Galerkin, and spectral vanishing viscosity methods respectively. In [15, 13] Kuznetsov type error estimates are given, but only for integrable Lévy measures or measures like (1.3) with $\lambda < 1$. Both of these results can be obtained through the framework of this paper. In [12] error estimates are given for all λ but with completely different methods. The general degenerate non-linear case is discretized in [11] (without error estimates) for symmetric α-stable Lévy measures and then in the most general case in the present paper.

Linear non-degenerate versions of (1.1) frequently arise in Finance, and the problem of solving these equations numerically has generated a lot of activity over the last decade. An introduction and overview of this activity can be found in the book [14], including numerical schemes based on truncation of the Lévy measure. We also mention the literature on fractional and nonlocal fully non-linear equations like e.g. the Bellman equation of optimal control theory. Such equations have been intensively studied over the last decade using viscosity solution methods, including initial results on numerical methods and error analysis. We refer e.g. [5, 8, 23] and references therein for an overview and the most general results in that direction. In fact, ideas from that field has been essential in the development of the entropy solution theory of equations like (1.1), and the construction of monotone numerical methods of this paper parallels the one in [8]. However the structure of the two classes of equations along with their mathematical and numerical analysis are very different.

This paper is organized as follows. In Section 2 we recall the entropy formulation and well-posedness results for (1.1) of [11] and the Kuznetsov type lemma derived in [2]. We present the numerical method in Section 3. There we focus on the case of no convection ($f \equiv 0$) to simplify the exposition and focus on new ideas. In Section 4 we prove several auxiliary properties of the numerical method which will be useful in the following sections. We establish existence, uniqueness, and a priori estimates for the solutions of the numerical method in Section 5. The general Kuznetsov type theory for deriving error estimates is presented in Section 6, where it is also used to establish a rate of convergence for equations with fractional Lévy measures, i.e. (1.5) holds. In Section 7 we extend all the results considered so far to general convection-diffusion equations of the form (1.1) with $f \not\equiv 0$. Finally, we give the proof of the main error estimate Theorem 6.1 in Section 8.

2. Preliminaries

In this section we briefly recall the entropy formulation for equations of the form (1.1) introduced in [11], and the new Kuznetsov type of lemma established in [2]. Let $\eta(u,k) = |u-k|$, $\eta'(u,k) = \operatorname{sgn}(u-k)$, $q_l(u,k) = \eta'(u,k)\,(f_l(u) - f_l(k))$ for $l = 1,\dots,d$, and write the nonlocal operator $\mathcal{L}^\mu[\phi]$ as

$$\mathcal{L}^\mu_r[\phi] + \mathcal{L}^{\mu,r}[\phi] + \gamma^{\mu,r}\cdot\nabla\phi,$$

where

$$\begin{aligned}
\mathcal{L}^\mu_r[\phi](x) &= \int_{0<|z|\le r} \phi(x+z) - \phi(x) - z\cdot\nabla\phi(x)\mathbf{1}_{|z|\le 1}\ \mathrm{d}\mu(z),\\
\mathcal{L}^{\mu,r}[\phi](x) &= \int_{|z|>r} \phi(x+z) - \phi(x)\ \mathrm{d}\mu(z),\\
\gamma^{\mu,r}_l &= -\int_{|z|>r} z_l\mathbf{1}_{|z|\le 1}\ \mathrm{d}\mu(z), \qquad l = 1,\dots,d.
\end{aligned}$$

We also define μ^* by $\mu^*(B) = \mu(-B)$ for all Borel sets $B \not\ni 0$. Let us recall that

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathcal{L}^\mu[\psi](x)\ \mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\,\mathcal{L}^{\mu^*}[\varphi](x)\ \mathrm{d}x$$

for all smooth $L^\infty\cap L^1$ functions φ,ψ, cf. [2, 11].

Definition 2.1. (Entropy solutions) *A function $u \in L^\infty(Q_T)\cap C([0,T];L^1(\mathbb{R}^d))$ is an entropy solution of* (1.1) *if, for all $k\in\mathbb{R}$, $r>0$, and test functions $0\le\varphi\in C^\infty_c(\mathbb{R}^d\times[0,T])$,*

$$\begin{aligned}
&\int_{Q_T} \eta(u,k)\,\partial_t\varphi + \Big(q(u,k) + \gamma^{\mu^*,r}\Big)\cdot\nabla\varphi + \eta(A(u),A(k))\,\mathcal{L}^{\mu^*}_r[\varphi]\\
&\qquad\qquad\qquad + \eta'(u,k)\,\mathcal{L}^{\mu,r}[A(u)]\,\varphi\ \mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_{\mathbb{R}^d} \eta(u(x,T),k)\,\varphi(x,T)\ \mathrm{d}x + \int_{\mathbb{R}^d} \eta(u_0(x),k)\,\varphi(x,0)\ \mathrm{d}x \ge 0.
\end{aligned} \tag{2.1}$$

Note that $\gamma^{\mu,r}_l \equiv 0$ when the Lévy measure μ is symmetric, i.e. when $\mu^*\equiv\mu$. From [11] we now have the following well-posedness result.

Theorem 2.1. (Well-posedness) *Assume (A.1) – (A.4) hold. Then there exists a unique entropy solution u of* (1.1) *such that*

$$u \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty(0,T; BV(\mathbb{R}^d)),$$

and the following a priori estimates hold

$$\begin{aligned}
&\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)},\\
&\|u(\cdot,t)\|_{L^1(\mathbb{R}^d))} \le \|u_0\|_{L^1(\mathbb{R}^d)},\\
&|u(\cdot,t)|_{BV(\mathbb{R}^d)} \le |u_0|_{BV(\mathbb{R}^d)},\\
&\|u(\cdot,t) - u(\cdot,s)\|_{L^1(\mathbb{R}^d)} \le \sigma(|t-s|),
\end{aligned}$$

for all $t, s \in [0,T]$ *where*

$$\sigma(r) = \begin{cases} c\,r & \text{if } \int_{|z|>0} |z| \wedge 1 \;\mathrm{d}\mu(z) < \infty, \\ c\,r^{\frac{1}{2}} & \text{otherwise.} \end{cases}$$

Moreover, if also (1.5) *holds, then*

$$\sigma(r) = \begin{cases} c\,r & \text{if } \lambda \in (0,1), \\ c\,|r \ln r| & \text{if } \lambda = 1, \\ c\,r^{\frac{1}{\lambda}} & \text{if } \lambda \in (1,2). \end{cases}$$

The last a priori estimate is slightly more general then the one in [11], and follows e.g. in the limit from the estimates in Lemmas 5.3 and 5.4. We now recall the new Kuznetsov type of lemma established in [2]. Let

$$\omega \in C_c^\infty(\mathbb{R}), \quad 0 \le \omega \le 1, \quad \omega(\tau) = 0 \text{ for all } |\tau| > 1, \quad \text{and} \quad \int_{\mathbb{R}} \omega(\tau)\,\mathrm{d}\tau = 1,$$

and define $\omega_\delta(\tau) = \frac{1}{\delta}\,\omega\left(\frac{\tau}{\delta}\right)$, $\Omega_\epsilon(x) = \omega_\epsilon(x_1)\cdots\omega_\epsilon(x_d)$, and

$$\varphi^{\epsilon,\delta}(x,y,t,s) = \Omega_\epsilon(x-y)\,\omega_\delta(t-s)$$

for $\epsilon, \delta > 0$. We also need

$$\mathcal{E}_\delta(v) = \sup_{\substack{|t-s|<\delta \\ t,s\in[0,T]}} \|v(\cdot,t) - v(\cdot,s)\|_{L^1(\mathbb{R}^d)}. \tag{2.2}$$

In the following we let $\mathrm{d}w = \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$ and $C_T \ge 0$ be a constant depending on time and the initial data u_0 that may change from line to line.

Lemma 2.2. (Kuznetsov type of lemma) *Assume (A.1) – (A.4) hold. Let u be the entropy solution of* (1.1) *and v be any function in* $L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^d)) \cap$

$L^\infty(0,T;BV(\mathbb{R}^d))$ *with* $v(\cdot\,,0)=v_0(\cdot)$. *Then, for any* $\epsilon, r>0$ *and* $0<\delta<T$,

$$
\begin{aligned}
&\|u(\cdot,T)-v(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq \|u_0-v_0\|_{L^1(\mathbb{R}^d)} + C\left(\epsilon+\mathcal{E}_\delta(u)\vee\mathcal{E}_\delta(v)\right)\\
&\quad -\iint_{Q_T}\iint_{Q_T}\eta(v(x,t),u(y,s))\,\partial_t\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&\quad -\iint_{Q_T}\iint_{Q_T} q(v(x,t),u(y,s))\cdot\nabla_x\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&\quad +\iint_{Q_T}\iint_{Q_T}\eta(A(v(x,t)),A(u(y,s)))\,\mathcal{L}_r^{\mu^*}[\varphi^{\epsilon,\delta}(x,\cdot,t,s)](y)\;\mathrm{d}w\\
&\quad -\iint_{Q_T}\iint_{Q_T}\eta'(v(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\,\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&\quad -\iint_{Q_T}\iint_{Q_T}\eta(A(v(x,t)),A(u(y,s)))\,\gamma^{\mu^*,r}\cdot\nabla_x\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&\quad +\iint_{Q_T}\int_{\mathbb{R}^d}\eta(v(x,T),u(y,s))\,\varphi^{\epsilon,\delta}(x,T,y,s)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad -\iint_{Q_T}\int_{\mathbb{R}^d}\eta(v_0(x),u(y,s))\,\varphi^{\epsilon,\delta}(x,0,y,s)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

The proof is given in [2]. The original result of result of Kuznetsov in [28] is a special case when $\mu=0$ (or $A=0$).

3. The numerical method

In this section we derive our numerical method. Here and in the following sections we focus on the case $f\equiv 0$ to simplify the exposition and focus on the new ideas. The general case $f\not\equiv 0$ will then be treated at the end, in Section 7.

We will consider uniform space/time grids given by $x_\alpha=\alpha\,\Delta x$ for $\alpha\in\mathbb{Z}^d$ and $t_n=n\,\Delta t$ for $n=0,\dots,N=\frac{T}{\Delta t}$. We also use the following rectangular subdivisions of space

$$R_\alpha = x_\alpha+\Delta x\,(0,1)^d\quad\text{for}\quad\alpha\in\mathbb{Z}^d.$$

We start by discretizing the nonlocal operator, replacing the measure μ by the bounded truncated measure $\mathbf{1}_{|z|>\frac{\Delta x}{2}}(z)\mu$ and the gradient by a numerical gradient

$$\hat{D}_{\Delta x}=(\hat{D}_1,\cdots,\hat{D}_d),\tag{3.1}$$

where $\hat{D}_l\equiv D_l^\gamma$ are upwind finite difference operators defined by

$$D_l^\gamma\phi(x)=\begin{cases} D_l^+\phi(x):=\dfrac{\phi(x+\Delta x\,e_l)-\phi(x)}{\Delta x} & \text{for } \gamma_l^{\mu,\frac{\Delta x}{2}}>0,\\[2mm] D_l^-\phi(x):=\dfrac{\phi(x)-\phi(x-\Delta x\,e_l)}{\Delta x} & \text{otherwise.}\end{cases}\tag{3.2}$$

Here $e_1,\dots,e_d$ is the standard basis of $\mathbb{R}^d$. This gives an approximate nonlocal operator

$$
\begin{aligned}
&\hat{\mathcal{L}}^\mu[A(\phi)](x)\\
&=\int_{|z|>\frac{\Delta x}{2}} A(\phi(x+z))-A(\phi(x))\,\mathrm{d}\mu(z)+\gamma^{\mu,\frac{\Delta x}{2}}\cdot\hat{D}_{\Delta x}A(\phi(x)),
\end{aligned}\tag{3.3}
$$

which is monotone by upwinding and non-singular since the truncated measure is bounded.

A semidiscrete approximation of (1.1) with $f\equiv 0$ is then obtained by solving the approximate equation

$$\partial_t u=\hat{\mathcal{L}}^\mu[A(u)],\tag{3.4}$$

by a finite volume method on the spacial subdivision $\{R_\alpha\}_\alpha$. I.e. for each t, we look for piecewise constant approximate solution

$$U(x,t) = \sum_{\beta\in\mathbb{Z}^d} U_\beta(t)\,\mathbf{1}_{R_\beta}(x),$$

that satisfy (3.4) in weak form with $\frac{1}{\Delta x^d}\mathbf{1}_{R_\beta}$ as test functions: For every $\alpha \in \mathbb{Z}^d$,

$$\frac{1}{\Delta x^d}\int_{R_\alpha} \partial_t U\,dx = \frac{1}{\Delta x^d}\int_{R_\alpha} \hat{\mathcal{L}}^\mu[A(U)]\,dx.$$

Finally we discretize in time by replacing ∂_t by backward or forward differences $D^\pm_{\Delta t}$ and $U_\alpha(t)$ by a piecewise constant approximation U^n_α. The result is the implicit method

$$U^{n+1}_\alpha = U^n_\alpha + \Delta t\,\hat{\mathcal{L}}^\mu\langle A(U^{n+1})\rangle_\alpha, \tag{3.5}$$

and the explicit method

$$U^{n+1}_\alpha = U^n_\alpha + \Delta t\,\hat{\mathcal{L}}^\mu\langle A(U^{n})\rangle_\alpha \tag{3.6}$$

where

$$\hat{\mathcal{L}}^\mu\langle A(U^n)\rangle_\alpha = \frac{1}{\Delta x^d}\int_{R_\alpha}\hat{\mathcal{L}}^\mu[A(\bar U^n)](x)\,\mathrm{d}x,$$

and $\bar U^n(x) = \sum_{\beta\in\mathbb{Z}^d} U^n_\beta\,\mathbf{1}_{R_\beta}(x)$ is a piecewise constant x-interpolation of U. As initial condition for both methods we take

$$U^0_\alpha = \frac{1}{\Delta x^d}\int_{R_\alpha} u_0(x)\,\mathrm{d}x \qquad \text{for all} \qquad \alpha\in\mathbb{Z}^d.$$

Lemma 3.1.

$$\hat{\mathcal{L}}^\mu\langle A(U^n)\rangle_\alpha = \sum_{\beta\in\mathbb{Z}} G^\alpha_\beta\,A(U^n_\beta)$$

with $G^\alpha_\beta = G_{\alpha,\beta} + G^{\alpha,\beta}$ *and*

$$\begin{aligned} G_{\alpha,\beta} &= \frac{1}{\Delta x^d}\int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x,\\ G^{\alpha,\beta} &= \sum_{l=1}^d \gamma_l^{\mu,\frac{\Delta x}{2}}\frac{1}{\Delta x^d}\int_{R_\alpha} D^\gamma_l\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x. \end{aligned} \tag{3.7}$$

Remark 3.2. $G_{\alpha,\beta}$ is a Toeplitz matrix (cf. Lemma 4.1 (b)) while $G^{\alpha,\beta}$ is a tridiagonal matrix. When the measure μ is symmetric, then $G_{\alpha,\beta}$ is symmetric and $G^{\alpha,\beta}=0$.

Proof. Since

$$A(\bar U(x)) = \sum_{\beta\in\mathbb{Z}^d} A(U_\beta)\,\mathbf{1}_{R_\beta}(x) \qquad\text{and}\qquad \hat D_{\Delta x}\bar U(x) = \sum_{\beta\in\mathbb{Z}^d} U_\beta\,\hat D_{\Delta x}\mathbf{1}_{R_\beta}(x),$$

we find that

$$\begin{aligned}\Delta x^d \hat{\mathcal{L}}^\mu \langle A(U)\rangle_\alpha &= \int_{R_\alpha} \hat{\mathcal{L}}^\mu[A(\bar{U})](x)\ \mathrm{d}x\\ &= \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} A(\bar{U}(x+z)) - A(\bar{U}(x))\ \mathrm{d}\mu(z)\,\mathrm{d}x\\ &\qquad + \gamma^{\mu,\frac{\Delta x}{2}}\cdot \int_{R_\alpha}\sum_{\beta\in\mathbb{Z}^d} A(U_\beta)\,\hat{D}_{\Delta x}\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\\ &= \sum_{\beta\in\mathbb{Z}^d} A(U_\beta)\left(\int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x\right)\\ &\qquad + \sum_{\beta\in\mathbb{Z}^d} A(U_\beta)\left(\sum_{l=1}^d \gamma_l^{\mu,\frac{\Delta x}{2}}\int_{R_\alpha} D_l^\gamma \mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\right).\end{aligned}$$

The proof is complete. □

4. Properties of the numerical method

In this section we show that the numerical methods are conservative, monotone and consistent in the sense that certain cell entropy inequalities are satisfied. We start by a technical lemma summarizing the properties of the weights G^α_β defined in (3.7).

Lemma 4.1.
(a) $\sum_{\alpha\in\mathbb{Z}^d} G^\alpha_\beta = \sum_{\alpha\in\mathbb{Z}^d} G^\beta_\alpha = 0$ *for all* $\beta\in\mathbb{Z}^d$.
(b) $G^\beta_\alpha = G^{\beta+e_l}_{\alpha+e_l}$ *for all* $\alpha,\beta\in\mathbb{Z}^d$ *and* $l=1,\dots,d$.
(c) $G^\beta_\beta \le 0$ *and* $G^\alpha_\beta \ge 0$ *for* $\alpha\ne\beta$.
(d) There is $\bar{c}=\bar{c}(d,\mu)>0$ *such that* $G^\beta_\beta \ge -\frac{\bar{c}}{\hat{\sigma}_\mu(\Delta x)}$ *and where*

$$\hat{\sigma}_\mu(s) = \begin{cases} s & \text{when } \int |z|\wedge 1\, d\mu(z) < \infty,\\ s^2 & \text{otherwise.}\end{cases} \tag{4.1}$$

(e) If (1.5) *holds, then there is* $\bar{c}=\bar{c}(d,\lambda)>0$ *such that* $G^\beta_\beta \ge -\frac{\bar{c}}{\hat{\sigma}_\lambda(\Delta x)}$ *for*

$$\hat{\sigma}_\lambda(s) = \begin{cases} s^\lambda & \text{for } \lambda>1,\\ \frac{s}{|\ln s|} & \text{for } \lambda=1,\\ s & \text{for } \lambda<1.\end{cases} \tag{4.2}$$

Proof. (a) By the definitions of $G_{\alpha,\beta}, G^{\alpha,\beta}$ and Fubini's theorem,

$$\Delta x^d \sum_{\alpha\in\mathbb{Z}^d} G_{\alpha,\beta} = \int_{|z|>\frac{\Delta x}{2}}\Big(\int_{\mathbb{R}^d}\mathbf{1}_{R_\beta}(x+z)\ \mathrm{d}x - \int_{\mathbb{R}^d}\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\Big)\,\mathrm{d}\mu(z) = 0,$$

$$\Delta x^d \sum_{\alpha\in\mathbb{Z}^d} G^{\alpha,\beta} = \pm\sum_{l=1}^d \frac{\gamma_l^{\mu,\frac{\Delta x}{2}}}{\Delta x}\Big(\int_{\mathbb{R}^d}\mathbf{1}_{R_\beta}(x\pm\Delta x\, e_l)\mathrm{d}x - \int_{\mathbb{R}^d}\mathbf{1}_{R_\beta}(x)\mathrm{d}x\Big) = 0,$$

and, since $\sum_{\beta\in\mathbb{Z}^d}\mathbf{1}_{R_\beta}(x)\equiv 1$,

$$\Delta x^d \sum_{\beta\in\mathbb{Z}^d} G_{\alpha,\beta} = \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}}\Big(\sum_{\beta\in\mathbb{Z}^d}\mathbf{1}_{R_\beta}(x+z) - \sum_{\beta\in\mathbb{Z}^d}\mathbf{1}_{R_\beta}(x)\Big)\,\mathrm{d}\mu(z)\,\mathrm{d}x = 0,$$

$$\Delta x^d \sum_{\beta\in\mathbb{Z}^d} G^{\alpha,\beta} = \pm\sum_{l=1}^d \frac{\gamma_l^{\mu,\frac{\Delta x}{2}}}{\Delta x}\int_{R_\alpha}\Big(\sum_{\beta\in\mathbb{Z}^d}\mathbf{1}_{R_\beta}(x\pm\Delta x\, e_l) - \sum_{\beta\in\mathbb{Z}^d}\mathbf{1}_{R_\beta}(x)\Big)\mathrm{d}x = 0.$$

Therefore $\sum_{\alpha\in\mathbb{Z}^d} G^\alpha_\beta = \sum_{\alpha\in\mathbb{Z}^d}\left(G_{\alpha,\beta}+G^{\alpha,\beta}\right)=0$ and $\sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta=0$.

(b) Let $y = x + e_l$ and note that

$$\begin{aligned}\Delta x^d\ G_{\alpha,\beta} &= \int_{R_{\alpha+e_l}}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(y-\Delta x\, e_l+z)-\mathbf{1}_{R_\beta}(y-\Delta x\, e_l)\ \mathrm{d}\mu(z)\,\mathrm{d}y\\ &= \int_{R_{\alpha+e_l}}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_{\beta+e_l}}(y+z)-\mathbf{1}_{R_{\beta+e_l}}(y)\ \mathrm{d}\mu(z)\,\mathrm{d}y\\ &= \Delta x^d\ G_{\beta+e_l,\alpha+e_l}.\end{aligned}$$

In a similar fashion we get $G^{\alpha,\beta}=G^{\beta+e_l,\alpha+e_l}$.

(c) Note that

$$\Delta x^d\ G_{\beta,\beta} = \int_{R_\beta}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z)-1\ \mathrm{d}\mu(z)\ \mathrm{d}x\le 0.$$

while by the definition D^γ_l, see (3.2),

$$\Delta x^d\ G^{\beta,\beta} = -\sum_{l=1}^d \gamma_l^{\mu,\frac{\Delta x}{2}}\ \mathrm{sgn}\left(\gamma_l^{\mu,\frac{\Delta x}{2}}\right)\int_{R_\beta}\frac{\mathbf{1}_{R_\beta}(x)}{\Delta x}\ \mathrm{d}x\le 0.$$

For $\alpha\neq\beta$,

$$\Delta x^d\ G_{\alpha,\beta} = \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z)\ \mathrm{d}\mu(z)\,\mathrm{d}x\ge 0,$$

$G^{\alpha,\beta}=0$ for $\alpha\neq\beta\pm e_l$, and by the definition of D^γ_l,

$$\Delta x^d\ G^{\beta\pm e_l,\beta} = \sum_{l=1}^d \gamma_l^{\mu,\frac{\Delta x}{2}}\ \mathrm{sgn}\left(\gamma_l^{\mu,\frac{\Delta x}{2}}\right)\int_{R_\beta\pm e_l}\frac{\mathbf{1}_{R_\beta}(x\pm\Delta x\, e_l)}{\Delta x}\ \mathrm{d}x\ge 0.$$

Therefore $G^\beta_\beta = G_{\beta,\beta}+G^{\beta,\beta}\le 0$ and $G^\alpha_\beta = G_{\alpha,\beta}+G^{\alpha,\beta}\ge 0$ for $\alpha\neq\beta$.

(d) To find the lower bound on G^β_β we note that $\int_{R_\beta}\mathbf{1}_{R_\beta}(x+z)-\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\ge -\Delta x^d$, and hence

$$G_{\beta,\beta}\ge -\int_{|z|>\frac{\Delta x}{2}}\mathrm{d}\mu(z)\ge -\int_{|z|<1}\left(\left(\frac{|z|}{\frac{\Delta x}{2}}\right)^2\mathbf{1}_{|z|<1}(z)+\mathbf{1}_{|z|>1}(z)\right)\mathrm{d}\mu(z).$$

The bound then follows since

$$\Delta x\, G^{\beta,\beta}\ge -d\int_{\frac{\Delta x}{2}<|z|<1}|z|\ \mathrm{d}\mu(z)\ge -d\int_{0<|z|<1}\frac{|z|^2}{\frac{\Delta x}{2}}\ \mathrm{d}\mu(z).$$

When $\int |z|\wedge 1\, d\mu(z)<\infty$, the corresponding bound follows by a similar argument.

(e) When (1.5) hold we can estimate $G_{\beta,\beta}$ in the following way

$$\begin{aligned}G_{\beta,\beta} &\ge -\int_{\frac{\Delta x}{2}<|z|<1}\frac{|z|}{\frac{\Delta x}{2}}\frac{c_\lambda\mathrm{d}z}{|z|^{d+\lambda}}-\int_{|z|>1}\mathrm{d}\mu(z)\\ &= -\begin{cases} c_\lambda\frac{2}{\Delta x}\frac{\sigma_d}{1-\lambda}\left(1-\left(\frac{\Delta x}{2}\right)^{1-\lambda}\right)+C & \text{for } \lambda\neq 1,\\ -c_\lambda\frac{2}{\Delta x}\sigma_d\ln\frac{\Delta x}{2}+C & \text{for } \lambda=1.\end{cases}\end{aligned}$$

The last equality can be proved using polar coordinates, and σ_d is the surface area of the unit sphere in $\mathbb{R}^d$. Similarly we find that

$$\Delta x\, G^{\beta,\beta} \geq -d \int_{\frac{\Delta x}{2}<|z|<1} |z|\ \frac{c_\lambda \mathrm{d}z}{|z|^{d+\lambda}} = -dc_\lambda \begin{cases} \frac{\sigma_d}{1-\lambda}\Big(1-\big(\frac{\Delta x}{2}\big)^{1-\lambda}\Big) & \text{for } \lambda \neq 1, \\ -\sigma_d \ln \frac{\Delta x}{2} & \text{for } \lambda = 1, \end{cases}$$

and since $\Big(1-(\frac{\Delta x}{2})^{1-\lambda}\Big)$ is less than 1 or $(\frac{\Delta x}{2})^{1-\lambda}$ when $\lambda < 1$ or $\lambda > 1$ respectively (and when $\Delta x < 2$), the proof is complete. □

From the two facts that $G^\beta_\alpha \geq 0$ when $\alpha \neq \beta$ and $\mathrm{sgn}(u)A(u) = |A(u)|$, we now immediately get a Kato type inequality for the discrete nonlocal operator (3).

Lemma 4.2. *(Discrete Kato inequality) If $\{u_\alpha, v_\alpha\}_{\alpha\in\mathbb{Z}^d}$ are two bounded sequences, then*

$$\mathrm{sgn}(u_\alpha - v_\alpha) \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta (A(u_\beta) - A(v_\beta)) \leq \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta \left|A(u_\beta) - A(v_\beta)\right|.$$

From Lemma 4.1 it also follows that the explicit method (3.6) and the implicit method (3.5) are conservative and monotone, at least when the explicit method satisfies the following CFL condition:

$$\bar{c}L_A \frac{\Delta t}{\hat{\sigma}_\mu(\Delta x)} < 1 \qquad \text{where } \hat{\sigma}_\mu \text{ is defined in (4.1).} \tag{4.3}$$

Here $\bar{c}$ is defined in Lemma 4.1, and L_A denotes the Lipschitz constant of A. When the Lévy measure μ also satisfies (1.5), we have a weaker CFL condition

$$\bar{c}L_A \frac{\Delta t}{\hat{\sigma}_\lambda(\Delta x)} < 1 \qquad \text{where } \hat{\sigma}_\lambda \text{ is defined in (4.2).} \tag{4.4}$$

Proposition 4.3 (Conservative monotone schemes)**.**
(a) The implicit and explicit methods (3.5) *and* (3.6) *are conservative, i.e. for an l^1-solution U,*

$$\sum_\alpha U^n_\alpha = \sum_\alpha U^0_\alpha.$$

(b) The implicit method is monotone, i.e. if U and V solve (3.5)*, then*

$$U^n \leq V^n \qquad \Rightarrow \qquad U^{n+1} \leq V^{n+1} \quad \textit{for} \quad n \geq 0.$$

(c) If (4.3) *(or* (4.4) *and* (1.5)*) holds, then the explicit method* (3.6) *is monotone.*

Remark 4.4. The CFL condition (4.3) implies that $\frac{\Delta t}{\Delta x^2} \leq C$ in general (just as for the heat equation), and $\frac{\Delta t}{\Delta x} \leq C$ when $\int |z| \wedge 1\, d\mu(z) < \infty$. Condition (4.3) is sufficient for all equations considered in this paper. In real applications however, typically (1.5) holds, and the superior CFL condition (4.4) should be used.

Proof. (a) Sum (3.5) or (3.6) over α, change the order of summation, and use Lemma 4.1 (a):

$$\sum_{\alpha\in\mathbb{Z}^d} U^{n+1}_\alpha = \sum_{\alpha\in\mathbb{Z}^d} U^n_\alpha + \Delta t \sum_{\beta\in\mathbb{Z}^d} A(U_\beta)\Bigg(\sum_{\alpha\in\mathbb{Z}^d} G^\alpha_\beta\Bigg) = \sum_{\alpha\in\mathbb{Z}^d} U^n_\alpha.$$

(c) Let $T_\alpha[u] = u_\alpha + \Delta t \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta A(u_\beta)$, the right hand side of (3.6). By Lemma 4.1 (c), $G^\alpha_\beta \geq 0$ for $\alpha \neq \beta$ and hence

$$\partial_{u_\beta} T_\alpha[u] \geq 0 \quad \text{for} \quad \beta \neq \alpha.$$

Since A non-decreasing and $G^\alpha_\alpha \leq 0$, we use the lower bound on G^α_α in Lemma 4.1 (c) to find that

$$\partial_{u_\alpha} T_\alpha[u] = 1 + \Delta t\, G^\alpha_\alpha\, A'(u_\alpha) \geq 1 - \bar{c}L_A \frac{\Delta t}{\hat{\sigma}_\mu(\Delta x)},$$

which is positive by the CFL condition (4.3).
(b) The proof is similar to and easier than the proof of (c). □

We then turn to checking the consistency of the method, and to do that we write $G_{\alpha,\beta} = G^r_{\alpha,\beta} + G_{\alpha,\beta,r}$ and $G^{\alpha,\beta} = G^{\alpha,\beta,r} + G^{\alpha,\beta}_r$ for $r > 0$ where

$$\begin{aligned}
G^r_{\alpha,\beta} &= \frac{1}{\Delta x^d}\int_{R_\alpha}\int_{\frac{\Delta x}{2}<|z|\leq r} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x,\\
G_{\alpha,\beta,r} &= \frac{1}{\Delta x^d}\int_{R_\alpha}\int_{|z|>r} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x,\\
G^{\alpha,\beta,r} &= \frac{1}{\Delta x^d}\sum_{l=1}^d \gamma^{\mu,\frac{\Delta x}{2}}_{l,r}\int_{R_\alpha} D^{\gamma_r}_l \mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\\
&\text{for}\quad \gamma^{\mu,\frac{\Delta x}{2}}_{l,r} = -\int_{\frac{\Delta x}{2}<|z|\leq r} z_l \mathbf{1}_{|z|\leq 1}\ \mathrm{d}\mu(z),\\
G^{\alpha,\beta}_r &= \frac{1}{\Delta x^d}\sum_{l=1}^d \gamma^{\mu,r}_l\int_{R_\alpha} D^{\gamma^r}_l \mathbf{1}_{R_\beta}(x)\ \mathrm{d}x.
\end{aligned}$$

If $r < \frac{\Delta x}{2}$, we set $G^r_{\alpha,\beta} = 0 = G^{\alpha,\beta,r}$. We also define

$$G^{\beta,r}_\alpha = G^r_{\alpha,\beta} + G^{\alpha,\beta,r} \quad\text{and}\quad G^\beta_{\alpha,r} = G_{\alpha,\beta,r} + G^{\alpha,\beta}_r,$$

and note that Lemmas 4.1 and 4.2 obviously still holds with $G^{\beta,r}_\alpha$ or $G^\beta_{\alpha,r}$ replacing G^β_α.

Proposition 4.5. (Cell-entropy inequalities)
(a) If U is a solution of the implicit method (3.5)*, then, for all $r > 0$ and $k \in \mathbb{R}$,*

$$\begin{aligned}
\eta(U^{n+1}_\alpha, k) \leq \eta(U^n_\alpha, k) &+ \Delta t\sum_{\beta\in\mathbb{Z}^d} G^{\alpha,r}_\beta\,\eta(A(U^{n+1}_\beta), A(k))\\
&+ \Delta t\,\eta'(U^{n+1}_\alpha, k)\sum_{\beta\in\mathbb{Z}^d} G^\alpha_{\beta,r}\,A(U^{n+1}_\beta).
\end{aligned}\tag{4.5}$$

(b) Assume the CFL condition (4.3) *(or* (4.4) *and* (1.5)*) holds. If U is a solution of the explicit method* (3.6)*, then, for all $r > 0$ and $k \in \mathbb{R}$,*

$$\begin{aligned}
\eta(U^{n+1}_\alpha, k) \leq \eta(U^n_\alpha, k) &+ \Delta t\sum_{\beta\in\mathbb{Z}^d} G^{\alpha,r}_\beta\,\eta(A(U^n_\alpha), A(k))\\
&+ \Delta t\,\eta'(U^{n+1}_\alpha, k)\sum_{\beta\in\mathbb{Z}^d} G^\alpha_{\beta,r}\,A(U^n_\alpha).
\end{aligned}\tag{4.6}$$

Remark 4.6. In the cell-entropy inequality for the explicit method, the η'-term appears in the "wrong" time. In Section 6, we will see that this leads to worse error estimates for the explicit method than for the implicit method.

Remark 4.7 (Convergence to entropy solutions). Proposition 4.5 and a standard argument show that any $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$-convergent sequence of (interpolated) solutions $\bar{u}_{\Delta x}$ of (3.5) or (3.6), will converge to an entropy solution of (1.1). We refer to Theorem 3.9 in [22] and Section 4.2 in [11] for more details. Convergence to the entropy solution also follows from the error estimates of Section 6.

Proof. (a) By (3.5) we easily see that for any $k \in \mathbb{R}$,

$$\begin{aligned}
U^{n+1}_\alpha \vee k &\leq U^n_\alpha \vee k + \Delta t\,\mathbf{1}_{(k,+\infty)}(U^{n+1}_\alpha)\,\hat{\mathcal{L}}^\mu\langle A(U^{n+1})\rangle_\alpha,\\
U^{n+1}_\alpha \wedge k &\geq U^n_\alpha \wedge k + \Delta t\,\mathbf{1}_{(-\infty,k)}(U^{n+1}_\alpha)\,\hat{\mathcal{L}}^\mu\langle A(U^{n+1})\rangle_\alpha.
\end{aligned}$$

Subtracting and using $\eta(u,k) = |u-k|$ and $\eta'(u,k) = \operatorname{sgn}(u-k)$, we find that

$$\eta(U_\alpha^{n+1},k) \leq \eta(U_\alpha^n,k) + \Delta t\, \eta'(U_\alpha^{n+1},k)\, \hat{\mathcal{L}}^\mu \langle A(U^{n+1})\rangle_\alpha.$$

For any $r>0$, we use Lemmas 4.1 (a) and 4.2 with $G_\beta^{\alpha,r}$ replacing G_β^α to see that

$$\begin{aligned}
&\eta'(U_\alpha^{n+1},k) \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r}\, A(U_\beta^{n+1})\\
&= \eta'(U_\alpha^{n+1},k) \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r}\,(A(U_\beta^{n+1}) - A(k)) \qquad \left(\text{since } \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r} = 0\right)\\
&\leq \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r}\, \eta(A(U_\beta^{n+1}),A(k)).
\end{aligned}$$

The cell entropy inequality now follows from writing $G_\beta^\alpha = G_{\beta,r}^\alpha + G_\beta^{\alpha,r}$ and using the above inequalities.

(b) By (3.6) and monotonicity (Proposition 4.5 (c)) we obtain the following inequalities: For all $r>0$,

$$\begin{aligned}
U_\alpha^{n+1}\vee k &\leq U_\alpha^n \vee k + \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r}\, A(U_\beta^n\vee k)\\
&\quad + \Delta t\, \mathbf{1}_{(k,+\infty)}(U_\alpha^{n+1}) \sum_{\beta\in\mathbb{Z}^d} G_{\beta,r}^\alpha\, A(U_\beta^n),\\
U_\alpha^{n+1}\wedge k &\geq U_\alpha^n \wedge k + \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r}\, A(U_\beta^n\wedge k)\\
&\quad + \Delta t\, \mathbf{1}_{(-\infty,k)}(U_\alpha^{n+1}) \sum_{\beta\in\mathbb{Z}^d} G_{\beta,r}^\alpha\, A(U_\beta^n).
\end{aligned}$$

Since $\eta(A(U),A(k)) = A(U\vee k) - A(U\wedge k)$, the cell entropy inequality follows from subtracting the two inequalities. □

5. A priori estimates, existence, and uniqueness

In this section we state and prove several a priori estimates for the solutions of the numerical methods (3.5) and (3.6). In what follows, we will use different interpolants $\bar{u}$ of the solutions U_α^n of the schemes. For the implicit method (3.5) we take

$$\bar{u}(x,t) = U_\alpha^{n+1} \quad \text{for all } (x,t)\in R_\alpha\times(t_n,t_{n+1}], \tag{5.1}$$

while for the explicit method (3.6),

$$\bar{u}(x,t) = U_\alpha^{n} \quad \text{for all } (x,t)\in R_\alpha\times[t_n,t_{n+1}). \tag{5.2}$$

We now prove the following a priori estimates for $\bar{u}$:

$$\|\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}, \tag{5.3}$$

$$\|\bar{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}, \tag{5.4}$$

$$|\bar{u}(\cdot,t)|_{BV(\mathbb{R}^d)} \leq |u_0|_{BV(\mathbb{R}^d)}. \tag{5.5}$$

Lemma 5.1. (A priori estimates)
(a) If U solve (3.5) *and $\bar{u}$ is defined by* (5.1)*, then the a priori estimates* (5.3) – (5.5) *hold for all $t>0$.*

(b) Assume the CFL condition (4.3) *(or* (4.4) *and* (1.5)*) holds. If U solve* (3.6) *and $\bar{u}$ is defined by* (5.2)*, then the a priori estimates* (5.3) – (5.5) *hold for all $t>0$.*

Proof. Since the schemes are conservative and monotone, cf. Proposition 4.3, this is a standard result that essentially follows from the Crandall-Tartar Lemma. For explicit methods in part (b) we refer to e.g. Theorem 3.6 in [22] for the details.

We did not find a reference for implicit methods, so we give a proof of part (a) here. See also [17] for the case when A is linear. Let $u_\alpha = U_\alpha^{n+1}$, $h_\alpha = U_\alpha^n$, and write (3.5) as

$$u_\alpha - \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha \, A(u_\beta) = h_\alpha. \tag{5.6}$$

We prove (5.3). Multiply (5.6) by $\operatorname{sgn}(u_\alpha)$ and use Lemma 4.2 to get

$$|u_\alpha| - \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha \, |A(u_\beta)| \le |h_\alpha|,$$

which by Fubini's theorem and the fact that $\sum_{\alpha\in\mathbb{Z}^d} G_\beta^\alpha = 0$ implies that

$$\sum_{\alpha\in\mathbb{Z}^d} |u_\alpha| \le \sum_{\alpha\in\mathbb{Z}^d} |h_\alpha|.$$

By the definition of u_α, h_α and an iteration in n, it follows that

$$\sum_{\alpha\in\mathbb{Z}^d} |U_\alpha^n| \le \sum_{\alpha\in\mathbb{Z}^d} |U_\alpha^0|.$$

By (5.1), $\|\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} = \Delta x^d \sum_{\alpha\in\mathbb{Z}^d} |U_\alpha^n|$ for $t \in (t_n, t_{n+1}]$, and (5.3) follows.

To prove (5.5), we subtract two equations (5.6) evaluated at different points,

$$u_\alpha - u_{\alpha-e_l} - \Delta t \sum_{\beta\in\mathbb{Z}^d} \Big(G_\beta^\alpha \, A(u_\beta) - G_\beta^{\alpha-e_l} \, A(u_\beta) \Big) = h_\alpha - h_{\alpha-e_l}$$

and use the fact that $G_\beta^\alpha = G_{\beta+e_l}^{\alpha+e_l}$ to see that

$$u_\alpha - u_{\alpha-e_l} - \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha \Big(A(u_\beta) - A(u_{\beta-e_l}) \Big) = h_\alpha - h_{\alpha-e_l}.$$

Then we multiply by $\operatorname{sgn}(u_\alpha - u_{\alpha-e_l})$, use Lemma 4.2, and sum over α, to find that

$$\sum_{\alpha\in\mathbb{Z}^d} |u_\alpha - u_{\alpha-e_l}| \le \sum_{\alpha\in\mathbb{Z}^d} |h_\alpha - h_{\alpha-e_l}| \, .$$

The estimate (5.5) then follows by iteration and the definitions of $u_\alpha, h_\alpha, \bar{u}$.

It remains to prove (5.4). Note that since $\sum_\alpha |u_\alpha| < \infty$ by (5.3), there is an α_0 such that $\sup_\alpha u_\alpha = u_{\alpha_0}$. Moreover, the parabolic term is nonpositive at the maximum point: since $\sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha = 0$ and $\sum_{\beta\in\mathbb{Z}^d} |G_\beta^\alpha| < \infty$,

$$\sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha_0} \, A(u_\beta) = \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha_0} \Big(A(u_\beta) - A(u_{\alpha_0}) \Big) \le 0.$$

Then by the above inequality and (5.6),

$$\sup_{\alpha\in\mathbb{Z}^d} u_\alpha = u_{\alpha_0} \le u_{\alpha_0} - \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha_0} \, A(u_\beta) = h_{\alpha_0} \le \sup_{\alpha\in\mathbb{Z}^d} h_\alpha.$$

In a similar way we find that $\inf_{\alpha\in\mathbb{Z}^d} h_\alpha \le \inf_{\alpha\in\mathbb{Z}^d} u_\alpha$ and (5.4) follow from the definitions of $u_\alpha, h_\alpha, \bar{u}$ and an iteration in n. □

Lemma 5.2 (Global existence and uniqueness)**.**
(a) There exists a unique solution $U^n \in l^1$ of the implicit scheme (3.5) *for all $n \ge 0$.*

(b) Assume the CFL condition (4.3) *(or* (4.4) *and* (1.5)*) holds. Then there exists a unique solution $U^n \in l^1$ of the explicit scheme* (3.6) *for all $n \ge 0$.*

Note that $U^n \in l^1$ implies that $\bar{u}(\cdot,t) \in L^1(\mathbb{R}^d)$.

Proof. (a) Let $u_\alpha = U^{n+1}_\alpha$ and $h_\alpha = U^n_\alpha$, rewrite (3.5) as (5.6), define

$$T_\alpha[u] = u_\alpha - \epsilon\bigg(u_\alpha - \Delta t \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta\, A(u_\beta) - h_\alpha\bigg),$$

and let ϵ be such that

$$\epsilon\bigg(1 + L_A \bar{c}\frac{\Delta t}{\hat{\sigma}_\mu(\Delta x)}\bigg) < 1.$$

We first show that T_α is monotone, i.e. $u \le v$ implies $T_\alpha[u] \le T_\alpha[v]$. For $\alpha \neq \beta$, $G^\alpha_\beta \ge 0$ by Lemma 4.1, and hence since A non-decreasing,

$$\partial_{u_\beta} T_\alpha[u] \ge 0.$$

Moreover, since A non-decreasing and $-\frac{\bar{c}}{\hat{\sigma}_\mu(\Delta x)} \le G^\alpha_\alpha \le 0$,

$$\partial_{u_\alpha} T_\alpha[u] = 1 - \epsilon + \epsilon\,\Delta t\, G^\alpha_\alpha\, A'(u_\alpha) \ge 1 - \epsilon\bigg(1 + L_A \bar{c}\frac{\Delta t}{\hat{\sigma}_\mu(\Delta x)}\bigg)$$

which is positive by our choice of ϵ.

Since T is monotone and A is nondecreasing,

$$\begin{aligned}
&\sum_\alpha \Big(T_\alpha[u] - T_\alpha[v]\Big)^+ \le \sum_\alpha \Big(T_\alpha[u\vee v] - T_\alpha[v]\Big)\\
&= (1-\epsilon) \sum_{\alpha\in\mathbb{Z}^d} (u_\alpha \vee v_\alpha - v_\alpha) + \epsilon\,\Delta t \sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta \Big(A(u_\beta\vee v_\beta) - A(v_\beta)\Big)\\
&= (1-\epsilon) \sum_{\alpha\in\mathbb{Z}^d} (u_\alpha - v_\alpha)^+ + \epsilon\,\Delta t \sum_{\beta\in\mathbb{Z}^d}\Big(\sum_{\alpha\in\mathbb{Z}^d} G^\alpha_\beta\Big)\Big(A(u_\beta) - A(v_\beta)\Big)^+.
\end{aligned}$$

A similar estimate holds for $\sum_\alpha (T_\alpha[u]-T_\alpha[v])^-$, and since $\sum_{\alpha\in\mathbb{Z}^d} G^\alpha_\beta = 0$, we have shown that

$$\sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[u] - T_\alpha[v]| \le (1-\epsilon) \sum_{\alpha\in\mathbb{Z}^d} |u_\alpha - v_\alpha|.$$

So T_α is an l^1-contraction and Banach's fixed point theorem then implies that there exists a unique solution $\bar{u} \in l^1$ of $T_\alpha[\bar{u}] = \bar{u}_\alpha$ and hence also of (5.6).

(b) Existence follows by construction and the a priori estimates in Lemma 5.1. Uniqueness essentially follows by monotonicity and $\sum_\alpha G^\alpha_\beta = 0$: Assume two solutions U^n and V^n, subtract the two equations and multiply by $\operatorname{sgn}(U^n - V^n)$, and use the Kato inequality (Lemma 4.2) along with $\sum_\alpha G^\alpha_\beta = 0$ to show that $\sum_\alpha |U^n - V^n| \le \sum_\alpha |U^0 - V^0|$. □

We have the following regularity estimate in time:

Lemma 5.3. (Regularity in time)
(a) Assume (A.2) – (A.4) hold, and let U be a solution of the implicit method (3.5) *and $\bar{u}$ defined by* (5.1)*. Then*

$$\|\bar{u}(\cdot,s) - \bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \sigma_\mu(|s-t| + \Delta t)$$

for all $s,t > 0$, where

$$\sigma_\mu(r) = \begin{cases} r & \text{if } \int_{|z|>0} |z|\wedge 1\ \mathrm{d}\mu(z) < \infty,\\ \sqrt{r} & \text{otherwise.}\end{cases}$$

(b) Assume (A.2) *–* (A.4) *and* (4.3) *(or* (4.4) *and* (1.5)*) hold, and let* U *be a solution of the explicit method* (3.6) *and* $\bar{u}$ *defined by* (5.2)*. Then*

$$\|\bar{u}(\cdot,s)-\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)}\leq\sigma_\mu(|s-t|+\Delta t)$$

for all $s,t>0$*, where* σ_μ *is defined in (a).*

Proof. The two proofs are essentially identical, so we only do the proof for case (a).

1) By (3.5), we find that for any $x\in R_\alpha$,

$$U^n_\alpha-U^{n-1}_\alpha=\frac{\Delta t}{\Delta x^d}\int_{R_\alpha}\hat{\mathcal{L}}[A(\bar{U}^n)](x)\,dx.$$

Take a test function $0\leq\phi\in C_c^\infty$ and define $\phi_\alpha=\frac{1}{\Delta x^d}\int_{R_\alpha}\phi(y)\mathrm{d}y$ and $\bar{\phi}(x)=\sum_\alpha\phi_\alpha\mathbf{1}_{R_\alpha}(x)$. Multiply the equation by $\Delta x^d\phi_\alpha$ and sum over α to find that

$$\int_{\mathbb{R}^d}\bar{\phi}(x)(\bar{U}^n(x)-\bar{U}^{n-1}(x))\,\mathrm{d}x=\Delta t\int_{\mathbb{R}^d}\bar{\phi}(x)\hat{\mathcal{L}}[A(\bar{U}^n)](x)\,\mathrm{d}x,$$

where Let $\hat{\mathcal{L}}^*$ be the adjoint of $\hat{\mathcal{L}}$, then since $\bar{U}$ is constant over R_α,

$$\begin{aligned}&\int_{\mathbb{R}^d}\phi(x)(\bar{U}^n(x)-\bar{U}^{n-1}(x))\,\mathrm{d}x=\int_{\mathbb{R}^d}\bar{\phi}(x)(\bar{U}^n(x)-\bar{U}^{n-1}(x))\,\mathrm{d}x\\&=\Delta t\int_{\mathbb{R}^d}(\bar{\phi}(x)-\phi(x))\hat{\mathcal{L}}[A(\bar{U}^n)](x)\,\mathrm{d}x+\Delta t\int_{\mathbb{R}^d}\hat{\mathcal{L}}^*[\phi](x)A(\bar{U}^n)(x)\,\mathrm{d}x.\end{aligned}$$

2) Let ω_ε be an approximate unit, i.e. $\omega_\varepsilon(x)=\frac{1}{\varepsilon^d}\omega(\frac{x}{\varepsilon})$ where $0\leq\omega\in C_0^\infty$ and $\int_{\mathbb{R}^d}\omega\,dx=1$. Take $\phi(x)=\omega_\varepsilon(y-x)$ in the equation above and let $U^n_\varepsilon=\bar{U}^n*\omega_\varepsilon$:

$$\bar{U}^n_\varepsilon-\bar{U}^{n-1}_\varepsilon=\Delta t(\bar{\omega}_\varepsilon-\omega_\varepsilon)*\hat{\mathcal{L}}[A(\bar{U}^n)]+\Delta t\hat{\mathcal{L}}^*[\omega_\varepsilon]*A(\bar{U}^n).$$

By Fubini we then find that

$$\frac{1}{\Delta t}\|\bar{U}^n_\varepsilon-\bar{U}^{n-1}_\varepsilon\|_{L^1}\leq\|\bar{\omega}_\varepsilon-\omega_\varepsilon\|_{L^1}\|\hat{\mathcal{L}}[A(\bar{U}^n)]\|_{L^1}+\|\hat{\mathcal{L}}^*[\omega_\varepsilon]*A(\bar{U}^n)\|_{L^1}=I_1+I_2.$$

3) To estimate I_1, note that by a standard argument

$$\|\bar{\omega}_\varepsilon-\omega_\varepsilon\|_{L^1}\leq|\omega_\varepsilon|_{BV}\Delta x=\frac{c_\omega}{\varepsilon}\Delta x,$$

and then by the definition of $\hat{\mathcal{L}}$ in (3.3), Fubini, the $L^1\cap BV$ regularity of U^n (Lemma 5.1), and the regularity of A in (A.2),

$$\begin{aligned}&\|\hat{\mathcal{L}}[A(\bar{U}^n)]\|_{L^1}\\&=\int_{|z|>\frac{\Delta x}{2}}\int_{\mathbb{R}^d}A(\bar{U}^n(x+z))-A(\bar{U}^n(x))-z\cdot\hat{D}_{\Delta x}A(\bar{U}^n(x))\mathbf{1}_{|z|<1}\,\mathrm{d}x\,\mathrm{d}\mu(z)\\&\leq\int_{|z|>\frac{\Delta x}{2}}\Big(2|A(U^n)|_{BV}|z|\mathbf{1}_{|z|<1}+2\|A(U^n)\|_{L^1}\mathbf{1}_{|z|>1}\Big)\,\mathrm{d}\mu(z)\\&\leq C\int_{|z|>\frac{\Delta x}{2}}|z|\wedge1\,\mathrm{d}\mu(z)\leq\frac{C}{\Delta x}\int_{|z|>0}|z|^2\wedge1\,\mathrm{d}\mu(z).\end{aligned}$$

These estimates along with (A.3) shows that $I_1\leq C\varepsilon^{-1}$.

4) Then we estimate I_2. Note first that since $\hat{D}_{\Delta x} = D + (\hat{D}_{\Delta x} - D)$, we can use Taylor's formula to see that

$$\begin{aligned}&\phi(x+z) - \phi(x) - z\hat{D}_{\Delta x}\phi(x)\\&= \int_0^1 (1-s)z^T D^2\phi(x+sz)z\,\mathrm{d}s \pm \Delta x \sum_{i=1}^d z_i \int_0^1 (1-s)\phi_{x_i x_i}(x \pm s\Delta x)\,\mathrm{d}s.\end{aligned}$$

This identity along with the definition of $\hat{\mathcal{L}}^*$, repeated use of Fubini, and one integration by parts in x, then leads to

$$\begin{aligned}&\hat{\mathcal{L}}^*[\omega_\varepsilon] * A(\bar{U}^n)(x)\\&= -\int_0^1 \int_{\frac{\Delta x}{2}<|z|<1} \int_{\mathbb{R}^d} (1-s)D\omega_\varepsilon(x-y+sz)z \otimes z\, DA(\bar{U}^n(y))\,\mathrm{d}y\,\mathrm{d}\mu(z)\,\mathrm{d}s\\&\quad \mp \Delta x \sum_{i=1}^d \int_0^1 \int_{\frac{\Delta x}{2}<|z|<1} \int_{\mathbb{R}^d} (1-s)\partial_{x_i}\omega_\varepsilon(x-y\pm s\Delta x)z_i\, \partial_{x_i} A(\bar{U}^n(y))\,\mathrm{d}y\mathrm{d}\mu(z)\mathrm{d}s\\&\quad + \int_{\mathbb{R}^d} \int_{|z|>1} \Big(\omega_\varepsilon(x-y+z) - \omega_\varepsilon(x-y)\Big) A(\bar{U}^n(x))\,\mathrm{d}\mu(z)\,\mathrm{d}y.\end{aligned}$$

Here $DA(\bar{U}^n(y))\,\mathrm{d}y$ should be interpreted as a measure, and $\int |DA(\bar{U}^n(y))|\,\mathrm{d}y = \int \mathrm{d}|A(U^n)|(y) = |A(U^n)|_{BV}$. By Young's inequality for convolutions (Fubini in our case), we then find that

$$\begin{aligned}I_2 &\leq \frac{3}{2}|\omega_\varepsilon|_{BV}|A(\bar{U}^n)|_{BV} \int_{0<|z|<1} |z|^2\,\mathrm{d}\mu(z) + 2\|\omega_\varepsilon\|_{L^1}\|A(\bar{U}^n)\|_{L^1} \int_{|z|>1} \mathrm{d}\mu(z)\\&\leq C\varepsilon^{-1}.\end{aligned}$$

Here again we have used the properties and regularity of μ, A, $\bar{U}^n$, and ω_ε.

5) By steps 2) – 4) we can conclude that

$$\|\bar{U}^n_\varepsilon - \bar{U}^m_\varepsilon\|_{L^1} \leq \sum_{j=m+1}^n \|\bar{U}^j_\varepsilon - \bar{U}^{j-1}_\varepsilon\|_{L^1} \leq \frac{C}{\varepsilon}|n-m|\Delta t,$$

where the constant C does not depend on n or m. By the triangle inequality and standard BV-estimates, it then follows that

$$\begin{aligned}\|\bar{U}^n - \bar{U}^m\|_{L^1} &\leq \|\bar{U}^n - \bar{U}^n_\varepsilon\|_{L^1} + \|\bar{U}^n_\varepsilon - \bar{U}^m_\varepsilon\|_{L^1} + \|\bar{U}^m_\varepsilon - \bar{U}^m\|_{L^1}\\&\leq |\bar{U}^n|_{BV}\varepsilon + \frac{C}{\varepsilon}|n-m|\Delta t + |\bar{U}^m|_{BV}\varepsilon,\end{aligned}$$

and hence by taking $\varepsilon = C\sqrt{|n-m|\Delta t}$,

$$\|\bar{U}^n - \bar{U}^m\|_{L^1} \leq C\sqrt{|n-m|\Delta t}.$$

For the time-interpolated function $\bar{u}$ defined in (5.1), we then find the following estimate

$$\|\bar{u}(\cdot,t) - \bar{u}(\cdot,s)\|_{L^1} = \|\bar{U}^n - \bar{U}^m\|_{L^1} \leq C\sqrt{|n-m|\Delta t} \leq C\sqrt{|t-s|+\Delta t}.$$

The equality follows since for each t, s there are n, m such that $\bar{u}(x,t) = \bar{U}^n(x)$ and $\bar{u}(x,s) = \bar{U}^m(x)$. Moreover, by the definition of $\bar{u}$, $|n-m|\Delta t \leq |t-s| + \Delta t$.

It remains to prove a better estimate for the case when $\int |z| \wedge 1\,\mathrm{d}\mu(z) < \infty$. This proof is similar but much easier than the proof above, so we skip it. □

The time regularity result in Lemma 5.3 is not optimal for Levy operators $\mathcal{L}$ with order in the interval $[1,2)$. To get optimal results we need more detailed information on the Levy measure μ than merely assumption (A.3). We will now prove an improved time regularity result for fractional measures (1.5). In this result we will need the following CFL condition,

$$C\frac{\Delta t}{\Delta x^{1\vee\lambda}} < 1 \qquad \text{for} \qquad \lambda \in (0,2). \tag{5.7}$$

Lemma 5.4. (Time regularity for fractional measures) *Assume (A.2) – (A.4), and* (1.5) *hold.*
(a) If the CFL condition (5.7) *hold, U is a solution of the implicit method* (3.5) *and $\bar{u}$ its interpolation defined by* (5.1)*, then for all $s,t>0$,*

$$\|\bar{u}(\cdot,s)-\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \sigma_\lambda(|s-t|+\Delta t); \qquad \sigma_\lambda(\tau) = \begin{cases} \tau & \lambda<1,\\ \tau|\ln\tau| & \lambda=1,\\ \tau^{\frac{1}{\lambda}} & \lambda>1.\end{cases}$$

(b) If the CFL condition (4.4) *hold, U is a solution of the explicit method* (3.6) *and $\bar{u}$ its interpolation defined by* (5.2)*, then for all $s,t>0$,*

$$\|\bar{u}(\cdot,s)-\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \sigma_\lambda(|s-t|+\Delta t); \quad \sigma_\lambda(\tau) = \begin{cases} \tau & \lambda<1,\\ \tau^\alpha \text{ for any } \alpha\in(0,1) & \lambda=1,\\ \tau^{\frac{1}{\lambda}} & \lambda>1.\end{cases}$$

Note well that in this result we *need* the CLF condition also for the implicit scheme. The reason is that the time-regularity is linked through the equation to the approximate Δx-depending diffusion term as will be seen from the proof. For the implicit scheme, we can have better results for $\lambda=1$ since we can use the less restrictive CFL condition (5.7).

Proof. The result for $\lambda<1$ is a corollary to Lemma 5.3. The proof for $\lambda\geq 1$ is the same as the proof of Lemma 5.3, except that we use different estimates for I_1 and I_2 in step 2). From step 3) in that proof and (1.5) and a simple computation in polar coordinates, we get that

$$\begin{aligned} I_1 &\leq C\frac{\Delta x}{\varepsilon}\int_{|z|>\frac{\Delta x}{2}} |z|\wedge 1\,\mathrm{d}\mu(z) \leq C\frac{\Delta x}{\varepsilon}\left(\int_{\frac{\Delta x}{2}<|z|<1}\frac{|z|\,dz}{|z|^{d+\lambda}}+C\right)\\ &\leq \frac{C}{\varepsilon}\begin{cases}\Delta x+\Delta x^{2-\lambda} & \text{for } \lambda>1,\\ \Delta x-\Delta x\ln\Delta x & \text{for } \lambda=1.\end{cases}\end{aligned}$$

To estimate I_2, we use Taylor expansions and integration by parts to find that

$$\begin{aligned}
&\hat{\mathcal{L}}^*[\omega_\varepsilon]*A(\bar{U}^n)(x) =\\
&-\int_0^1\int_{\frac{\Delta x}{2}<|z|<\varepsilon}\int_{\mathbb{R}^d}(1-s)D\omega_\varepsilon(x-y+sz)z\otimes z\,DA(\bar{U}^n(y))\,\mathrm{d}y\,\mathrm{d}\mu(z)\,\mathrm{d}s\\
&-\int_0^1\int_{\varepsilon<|z|<1}\int_{\mathbb{R}^d}\Big(\omega_\varepsilon(x-y+sz)-\omega_\varepsilon(x-y)\Big)z\,DA(\bar{U}^n(y))\,\mathrm{d}y\,\mathrm{d}\mu(z)\,\mathrm{d}s\\
&\mp\Delta x\sum_{i=1}^d\int_0^1\int_{\frac{\Delta x}{2}<|z|<\varepsilon}\int_{\mathbb{R}^d}(1-s)\partial_{x_i}\omega_\varepsilon(x-y\pm s\Delta x)z_i\,\partial_{x_i}A(\bar{U}^n(y))\,\mathrm{d}y\mathrm{d}\mu(z)\mathrm{d}s\\
&\mp\sum_{i=1}^d\int_0^1\int_{\varepsilon<|z|<1}\int_{\mathbb{R}^d}\Big(\omega_\varepsilon(x-y\pm s\Delta x)-\omega_\varepsilon(x-y)\Big)z_i\,\partial_{x_i}A(\bar{U}^n(y))\,\mathrm{d}y\mathrm{d}\mu(z)\mathrm{d}s\\
&+\int_{\mathbb{R}^d}\int_{|z|>1}\Big(\omega_\varepsilon(x-y+z)-\omega_\varepsilon(x-y)\Big)A(\bar{U}^n(x))\,\mathrm{d}\mu(z)\,\mathrm{d}y.
\end{aligned}$$

Then by Fubini, the definition of ω_ε, and the change of variables $(x,z)\to(\varepsilon x,\varepsilon z)$,

$$\int_{\mathbb{R}^d}\int_{\frac{\Delta x}{2}<|z|<\varepsilon}|D\omega_\varepsilon(x+sz)||z|^2\,\mathrm{d}\mu(z)\le c_\lambda\varepsilon^{1-\lambda}\int_{\mathbb{R}^d}|D\omega|\,\mathrm{d}x\int_{0<|z|<1}\frac{|z|^2\,\mathrm{d}z}{|z|^{d+\lambda}}.$$

By similar estimates and Young's inequality for convolutions we find that

$$\begin{aligned}I_2&\le c_\lambda\varepsilon^{1-\lambda}|A(\bar U^n)|_{BV}\left(3|\omega|_{BV}\int_{0<|z|<1}\frac{|z|^2\,\mathrm{d}z}{|z|^{d+\lambda}}+4\|\omega\|_{L^1}\int_{1<|z|<\frac{1}{\varepsilon}}\frac{|z|\,\mathrm{d}z}{|z|^{d+\lambda}}\right)\\&\quad+2\|A(\bar U^n)\|_{L^1}\|\omega_\varepsilon\|_{L^1}\int_{|z|>1}\mathrm{d}\mu(z)\\&\le C\begin{cases}\varepsilon^{1-\lambda}+1, & \lambda>1,\\|\ln\varepsilon|+1, & \lambda=1.\end{cases}\end{aligned}$$

Note that the $\ln\varepsilon$-term comes from the integral over $1<|z|<\frac{1}{\varepsilon}$.

As in step 5) in the proof of Lemma 5.3, we then find that

$$\|\bar U^n-\bar U^m\|_{L^1}\le|\bar U^n|_{BV}\varepsilon+|n-m|\Delta t(I_1+I_2)+|\bar U^m|_{BV}\varepsilon.$$

To conclude, we assume that $\Delta x\le\varepsilon$ which means in particular that

$$I_1+I_2\le C\begin{cases}\varepsilon^{1-\lambda}+1, & \lambda>1,\\|\ln\varepsilon|+1, & \lambda=1.\end{cases}$$

When $\lambda>1$, the final result follows from taking $\varepsilon=c(|n-m|\Delta t)^{\frac{1}{\lambda}}$ and arguing as in the end of the proof of Lemma 5.3. Note that in view of the CFL conditions (4.4) and (5.7), the constant c can be chosen such that $\Delta x\le\varepsilon$. For $\lambda=1$, we can use $\varepsilon=c|n-m|\Delta t$ for the implicit method in view of (5.7), and by (4.4), $\varepsilon=c(|n-m|\Delta t)^{\alpha'}$ for any $\alpha'\in(0,1)$, will do the job for the explicit method. □

By the a priori estimates Lemma 5.1 and 5.3 and Kolmogorov's compactness theorem (cf. e.g. [22, Theorem 3.8]), we find subsequences of both methods (3.5) and (3.6) converging to some function u. The function u inherits all the a priori estimates of $\bar u$, and it will be the unique entropy solution of (1.1) by Remark 4.7. In short, we have the following result:

Theorem 5.5. (Compactness) *Assume (A.2) – (A.4) hold. If either*

(i) *U is the solution of the implicit method* (3.5) *and $\bar u$ defined by* (5.1)*, or*
(ii) *U is the solution of the explicit method* (3.6)*, $\bar u$ defined by* (5.2)*, and* (4.3) *(or* (4.4) *and* (1.5)*) also holds,*

then there is a subsequence of $\{\bar u\}_{\Delta x>0}$ converging in $C([0,T];L^1(\mathbb{R}^d))$ to the unique entropy solution u of (1.1) *as $\Delta x\to0$. Moreover,*

$$u\in L^\infty(Q_T)\cap C([0,T];L^1(\mathbb{R}^d))\cap L^\infty(0,T;BV(\mathbb{R}^d)).$$

Remark 5.6. This result provides a proof for the existence result Theorem 5.3 in [11] for $L^1\cap L^\infty\cap BV$ entropy solutions of (1.1), and then the general existence result in $L^1\cap L^\infty$ follows by a density argument using the L^1-contraction.

6. Error estimates

In this section we give different error estimates and convergence results for our schemes, estimates that are valid for general Levy measures and better estimates that holds for fractional measures satisfying (1.5). To give the general result, we need the following quantities:

$$
\begin{aligned}
I_1^{\epsilon,r} &= \frac{1}{\epsilon}\int_{|z|\le r} |z|^2 \,\mathrm{d}\mu(z),\\
I_2^{\epsilon,\delta,r} &= \left(\frac{\Delta x}{\epsilon}+\frac{\Delta t}{\delta}\right)\left(\int_{r<|z|\le 1} |z|\,\mathrm{d}\mu(z)+\int_{|z|>1}\mathrm{d}\mu(z)\right);\\
I_3^{r} &= \mathcal{E}_{\Delta t}(\bar u)\int_{|z|>r}\mathrm{d}\mu(z).
\end{aligned}
$$

Theorem 6.1. (Error estimates) *Assume (A.2) – (A.4) hold, and let u be the entropy solution of* (1.1)*.*

(a) Let U be a solution of the implicit method (3.5) *and $\bar u$ defined by* (5.1)*. Then for all $\epsilon>0$, $0<\delta<T$, and $\frac{\Delta x}{2}<r\le 1$,*

$$
\|u(\cdot,T)-\bar u(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C_T\left(\epsilon+\mathcal{E}_\delta(u)\vee\mathcal{E}_\delta(\bar u)+I_1^{\epsilon,r}+I_2^{\epsilon,\delta,r}\right), \tag{6.1}
$$

(b) Assume also (4.3) *holds, and let U be a solution of the explicit method* (3.6) *and $\bar u$ defined by* (5.2)*. Then for all $\epsilon>0$, $0<\delta<T$, and $\frac{\Delta x}{2}<r\le 1$,*

$$
\|u(\cdot,T)-\bar u(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C_T\left(\epsilon+\mathcal{E}_\delta(u)\vee\mathcal{E}_\delta(\bar u)+I_1^{\epsilon,r}+I_2^{\epsilon,\delta,r}+I_3^{r}\right), \tag{6.2}
$$

The proof of this result will be given in Section 8.

Corollary 6.2 (Convergence)**.** *Under the assumptions of Theorem 6.1, the solutions of the implicit method* (3.5) *and the explicit method* (3.6) *both converge to the unique entropy solution of* (1.1) *as $\Delta x,\Delta t\to 0$.*

Proof. The result follows from the error estimates of Theorem 6.1 by first sending $\Delta x,\Delta t\to 0$, then $r\to 0$, and finally $\varepsilon,\delta\to 0$. □

We will now see how Theorem 6.1 (along with Lemma 5.4) can be used to produce explicit rates of convergence for our scheme in the case of fractional measures satisfying (1.5). First we define

$$
\sigma_\lambda^{IM}(\tau)=\begin{cases} \tau^{\frac12} & \lambda\in(0,1),\\ \tau^{\frac12}|\log\tau| & \lambda=1,\\ \tau^{\frac{2-\lambda}{2}} & \lambda\in(1,2),\end{cases} \tag{6.3}
$$

and

$$
\sigma_\lambda^{EX}(\tau)=\begin{cases} \tau^{\frac12} & \lambda\in\left(0,\frac23\right],\\ \tau^{\frac{2-\lambda}{2+\lambda}} & \lambda\in(\frac23,1)\cup(1,2).\end{cases} \tag{6.4}
$$

Theorem 6.3. (Convergence rate for fractional measures) *Under the assumptions of Lemma 5.4 (including* (1.5) *and a CFL condition for the implicit scheme), for all $\lambda\in(0,2)$,*

$$
\|u(\cdot,T)-\bar u(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le \begin{cases} C_T\,\sigma_\lambda^{IM}(\Delta x) & \text{for the implicit method (3.5)},\\[2ex] C_T\,\sigma_\lambda^{EX}(\Delta x) & \text{for the explicit method (3.6)}.\end{cases}
$$

Note that the rate for the explicit method is worse due to the extra term I_3^r in Theorem 6.1.

Corollary 6.4 (Explicit scheme when $\lambda = 1$)**.** *Let the assumptions of Lemma 5.4 (b) hold with $\lambda = 1$ and let $\alpha \in (1,2)$ be arbitrary. If the stronger CLF condition $C\frac{\Delta t}{\Delta x^\alpha} < 1$ holds, then*

$$\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq C_T\, \sigma_\alpha^{EX}(\Delta x) \quad \textit{for the explicit method } (3.6).$$

Proof. Note that the CFL condition (4.4) is satisfied and that the assumption (1.5) holds with any $\lambda \in [1,2)$. Hence the result follows from the $\lambda > 1$ case in Theorem 6.3. □

Proof of Theorem 6.3. Let us first give the proof for the implicit method (3.5). First we note that by (1.5),

$$\int_{|z|\leq r} |z|^2 \, \mathrm{d}\mu(z) \leq c_\lambda \int_{|z|\leq r} \frac{|z|^2}{|z|^{d+\lambda}} \, \mathrm{d}z \leq O\left(r^{2-\lambda}\right) \quad \text{for all } \lambda \in (0,2),\ r \leq 1,$$

while

$$\int_{r<|z|\leq 1} |z| \, \mathrm{d}\mu(z) \leq c_\lambda \int_{r<|z|\leq 1} \frac{|z|}{|z|^{d+\lambda}} \, \mathrm{d}z = \begin{cases} O(1) & \text{if } \lambda \in (0,1), \\ O(|\ln r|) & \text{if } \lambda = 1, \\ O\left(r^{1-\lambda}\right) & \text{if } \lambda \in (1,2). \end{cases}$$

Using these estimates along with the CFL condition (5.7) and Lemma 5.4, we find that the estimate (6.1) in Theorem 6.1 takes the form

$$\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq C_T \begin{cases} \epsilon + \delta + \frac{r^{2-\lambda}}{\epsilon} + \left(\frac{\Delta x}{\epsilon} + \frac{\Delta x}{\delta}\right) & \text{if } \lambda \in (0,1), \\ \epsilon + \delta\,|\ln \delta| + \frac{r}{\epsilon} + |\ln r| \left(\frac{\Delta x}{\epsilon} + \frac{\Delta x}{\delta}\right) & \text{if } \lambda = 1, \\ \epsilon + \delta^{\frac{1}{\lambda}} + \frac{r^{2-\lambda}}{\epsilon} + r^{1-\lambda}\left(\frac{\Delta x}{\epsilon} + \frac{\Delta x^\lambda}{\delta}\right) & \text{if } \lambda \in (1,2). \end{cases}$$

The conclusion then follows by taking $r = \Delta x$ for all $\lambda \in (0,2)$, $\epsilon = \delta = \sqrt{\Delta x}$ for $\lambda \in (0,1]$, while $\epsilon = \Delta x^{\frac{2-\lambda}{2}}$ and $\delta = \Delta x^{\frac{\lambda}{2}}$ for $\lambda \in (1,2)$.

For the explicit method (3.6) we also need to take into account the extra I_3-term,

$$I_3^r = \sigma_\lambda(\Delta t) \underbrace{\int_{|z|>r} \mathrm{d}\mu(z)}_{O(r^{-\lambda})},$$

Lemma 5.4, and the slightly more restrictive CFL condition (4.4). The expression (6.2) in Theorem 6.1 then takes the form

$$\begin{aligned} &\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \\ &\leq C_T \begin{cases} \epsilon + \delta + \frac{r^{2-\lambda}}{\epsilon} + \left(\frac{\Delta x}{\epsilon} + \frac{\Delta x}{\delta}\right) + \frac{\Delta x}{r^\lambda} & \text{if } \lambda \in (0,1), \\ \epsilon + \delta^{\frac{1}{\lambda}} + \frac{r^{2-\lambda}}{\epsilon} + r^{1-\lambda}\left(\frac{\Delta x}{\epsilon} + \frac{\Delta x^\lambda}{\delta}\right) + \frac{\Delta x}{r^\lambda} & \text{if } \lambda \in (1,2). \end{cases} \end{aligned}$$

We minimize two and two terms and take the maximum minimizers, first w.r.t. ε and δ and then w.r.t. r,

$$\begin{aligned} &\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \\ &\leq C_T \begin{cases} r^{\frac{2-\lambda}{2}} + \Delta x^{\frac{1}{2}} + \frac{\Delta x}{r^\lambda}, & \text{if } \lambda \in (0,1) \\ r^{\frac{2-\lambda}{2}} + r^{\frac{1-\lambda}{2}}\Delta x^{\frac{1}{2}} + r^{\frac{1-\lambda}{1+\lambda}}\Delta x^{\frac{\lambda}{1+\lambda}} + \frac{\Delta x}{r^\lambda} & \text{if } \lambda \in (1,2), \end{cases} \\ &\leq C_T \begin{cases} \Delta x^{\frac{1}{2}} + \Delta x^{\frac{2-\lambda}{2+\lambda}} & \text{if } \lambda \in (0,1), \\ \Delta x^{\frac{2-\lambda}{2}} + \Delta x^{\frac{2-\lambda}{3-\lambda}} + \Delta x^{\frac{2-\lambda}{2+\lambda}} & \text{if } \lambda \in (1,2). \end{cases} \end{aligned}$$

The final result follows since $\frac{2-\lambda}{3-\lambda} > \frac{2-\lambda}{2+\lambda}$ for $\lambda \in (\frac{1}{2}, 2)$ and $\frac{2-\lambda}{2+\lambda} < \frac{1}{2}$ for $\lambda \in (\frac{2}{3}, 2)$. □

Remark 6.5. The rates can not be improved by taking a different truncation of the singularity, i.e. replacing in the method

$$G_{\alpha,\beta} = \frac{1}{\Delta x^d} \int_{R_\alpha} \int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x) \,\mathrm{d}\mu(z)\,\mathrm{d}x$$

by

$$G_{\alpha,\beta} = \frac{1}{\Delta x^d} \int_{R_\alpha} \int_{|z|>\rho_\lambda(\Delta x)} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x) \,\mathrm{d}\mu(z)\,\mathrm{d}x.$$

The reason is that the function ρ_λ that minimize the error expression

$$\epsilon + \delta + \frac{\rho_\lambda^{2-\lambda}(\Delta x)}{\epsilon} + \rho_\lambda^{1-\lambda}(\Delta x)\left(\frac{\Delta x}{\epsilon} + \frac{\Delta x}{\delta}\right),$$

is always $\rho_\lambda(\Delta x) = O(\Delta x)$!

Remark 6.6. We believe that the rates for the implicit schemes are optimal, at least when there are nonlinear convection terms in the equation (i.e. when $f \neq 0$ in (1.1), see Section 7). But we have not found analytical examples confirming this, nor have we been able to observe the above rates in preliminary, but probably too crude, numerical tests. Maybe it is not straight forward to construct analytical or numerical examples confirming the optimality of the rates. We leave it as a challenge for people with more experience in realizing numerical schemes to test the optimality numerically.

7. Convection-diffusion equations

In this section we discuss how to extend the results established in the previous sections to the case $f \neq 0$. Note that all the arguments needed to handle the additional f-term are well-known. We consider the following numerical methods

$$U_\alpha^{n+1} = U_\alpha^n + \Delta t \sum_{l=1}^d D_l^- \hat{f}_l(U_\alpha^{n+1}, U_{\alpha+e_l}^{n+1}) + \Delta t\, \hat{\mathcal{L}}^\mu \langle A(U^{n+1})\rangle_\alpha, \quad \text{(implicit)} \tag{7.1}$$

$$U_\alpha^{n+1} = U_\alpha^n + \Delta t \sum_{l=1}^d D_l^- \hat{f}_l(U_\alpha^{n}, U_{\alpha+e_l}^{n}) + \Delta t\, \hat{\mathcal{L}}^\mu \langle A(U^{n+1})\rangle_\alpha, \quad \text{(expl-impl)} \tag{7.2}$$

$$U_\alpha^{n+1} = U_\alpha^n + \Delta t \sum_{l=1}^d D_l^- \hat{f}_l(U_\alpha^{n}, U_{\alpha+e_l}^{n}) + \Delta t\, \hat{\mathcal{L}}^\mu \langle A(U^{n})\rangle_\alpha, \quad \text{(explicit)} \tag{7.3}$$

where

(i) $D_l^- U_\alpha = \frac{1}{\Delta x}(U_\alpha - U_{\alpha-e_l})$ and $\{e_l\}_l$ is the standard basis of $\mathbb{R}^d$, and

(ii) $\hat{f} = (\hat{f}_1, \dots, \hat{f}_d)$ is a consistent (i.e. $\hat{f}(u,u) = f(u)$), Lipschitz continuous numerical flux which is non-decreasing w.r.t. the first variable and non-increasing w.r.t. the second one.

Remark 7.1. Some examples of numerical fluxes $\hat{f}$ satisfying (ii) are the well-known Lax-Friedrichs flux, the Godunov flux, and the Engquist-Osher flux, cf. e.g. [26].

For the schemes (7.2) and (7.3), we also need the CFL conditions

$$2d\, L_F \frac{\Delta t}{\Delta x} + \bar{c} L_A \frac{\Delta t}{\hat{\sigma}_\mu(\Delta x)} < 1 \qquad \text{and} \qquad 2d L_F \frac{\Delta t}{\Delta x} < 1 \tag{7.4}$$

respectively (compare with (4.3)), where $\hat{\sigma}_\mu$ is defined in (4.1) and L_F is the Lipschitz constant of $\hat{f}$. Then the all the a priori estimates and other results of Section

5 continue to hold for the new schemes, and we still have compactness via Kolmogorov's theorem. The modifications needed to identify the any limit as the unique entropy solution of (1.1) are standard and can be found e.g. in Chapter 3 in [22], and hence the convergence of the methods (7.1)–(7.3) follows.

We will now give the statement of the result of Theorem 6.1 that is valid for the current setting where $f \neq 0$. To do so we reuse the quantities $I_1^{\epsilon,r}$ and I_3^r of section 6, but redefine $I_2^{\epsilon,\delta,r}$ as follows

$$I_2^{\epsilon,\delta,r} = \left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\right)\left(1 + \int_{r<|z|\leq 1} |z|\,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z)\right).$$

Theorem 7.2. (Error estimates) *Assume (A.1) – (A.4) hold, and let u be the entropy solution of* (1.1)*.*

(a) Let U be a solution of (7.1) or (7.2) *and $\bar{u}$ defined by* (5.1)*. For* (7.2) *we also need the second CLF condition in* (7.4)*. Then for all $\epsilon > 0$, $0 < \delta < T$, and $\frac{\Delta x}{2} < r \leq 1$,*

$$\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq C_T\left(\epsilon + \mathcal{E}_\delta(u) \vee \mathcal{E}_\delta(\bar{u}) + I_1^{\epsilon,r} + I_2^{\epsilon,\delta,r}\right).$$

(b) Assume also that the first CFL condition in (7.4) *holds, and let U be a solution of* (7.3) *and $\bar{u}$ defined by* (5.2)*. Then for all $\epsilon > 0$, $0 < \delta < T$, and $\frac{\Delta x}{2} < r \leq 1$,*

$$\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq C_T\left(\epsilon + \mathcal{E}_\delta(u) \vee \mathcal{E}_\delta(\bar{u}) + I_1^{\epsilon,r} + I_2^{\epsilon,\delta,r} + I_3^r\right).$$

The proof is essentially equal to the proof of Theorem 6.1 augmented by standard Kuznetsov type computations to handle the f-term, cf. e.g. [22, Example 3.14]. We skip it.

Remark 7.3. It is easy to see that the contribution to the error from the discretization of the f-term is always less or of the same order as the contributions of the other terms. In particular, for fractional measures (1.5), we immediately get that the schemes satisfy the error estimate of Theorem 6.3 with modulus σ_λ^{IM} for (7.1) and (7.2) and modulus σ_λ^{EX} for (7.3).

8. The proof of Theorem 6.1

Proof of Theorem 6.1 for the implicit method (3.5)*.*

1. We use Lemma 2.2 to compare the solution of the scheme to the exact solution. In the resulting inequality, we introduce the scheme via the time derivative and the initial/final terms. To do this, we use integration by parts on each interval (t_n, t_{n+1}) and summation by parts to get discrete time derivatives on $\bar{u}$ so that we can use the cell entropy inequality (4.5). We get that (remember the definition of $\bar{u}$)

$$\begin{aligned}
&-\iint_{Q_T}\iint_{Q_T} \eta(\bar{u}(x,t),u(y,s))\,\partial_t\varphi^{\epsilon,\delta}(x,y,t,s)\mathrm{d}w + \text{initial and final terms}\\
&= \iint_{Q_T}\sum_{n=0}^{N-1}\sum_{\alpha\in\mathbb{Z}^d}\Big(\eta(U_\alpha^{n+1},u(y,s)) - \eta(U_\alpha^n,u(y,s))\Big)\int_{R_\alpha}\varphi^{\epsilon,\delta}(x,y,t_{n+1},s)\,\mathrm{d}x\,\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Let $\bar{\varphi}^{\epsilon,\delta} = \bar{\varphi}^{\epsilon,\delta}(x,y,t,s)$ be the function which for each $(y,s)\in Q_T$ is defined by

$$\varphi_\alpha^n = \frac{1}{\Delta x^d}\int_{R_\alpha}\varphi^{\epsilon,\delta}(x,y,t_n,s)\,\mathrm{d}x \quad \text{for} \quad x\in R_\alpha,\ t\in(t_{n-1},t_n],$$

and use above equation along with the cell entropy inequality (4.5) and Lemma 3.1 to write the inequality of Lemma 2.2 in the following way

$$\begin{aligned}
\|u(\cdot,T)-\bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} &\leq C_T\left(\Delta x+\epsilon+\mathcal{E}_\delta(u)\vee\mathcal{E}_\delta(v)\right)\\
&+\underbrace{\iint_{Q_T}\iint_{Q_T}\eta(A(\bar{u}(x,t)),A(u(y,s)))\,\mathcal{L}_r^{\mu^*}[\varphi^{\epsilon,\delta}(x,\cdot,t,s)](y)\;\mathrm{d}w}_{H_1}\\
&+\underbrace{\iint_{Q_T}\iint_{Q_T}\eta(A(\bar{u}(x,t)),A(u(y,s)))\,\hat{\mathcal{L}}_r^{\mu^*}[\bar{\varphi}^{\epsilon,\delta}(\cdot,y,t,s)](x)\;\mathrm{d}w}_{H_2}\\
&+\underbrace{\iint_{Q_T}\iint_{Q_T}\eta'(\bar{u}(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar{u}(\cdot,t))](x)\,(\bar{\varphi}^{\epsilon,\delta}-\varphi^{\epsilon,\delta})(x,y,t,s)\;\mathrm{d}w}_{H_3}\\
&+\underbrace{\iint_{Q_T}\iint_{Q_T}\eta(A(\bar{u}(x,t)),A(u(y,s)))\,\gamma^{\mu^*,r}\cdot(\hat{D}\bar{\varphi}^{\epsilon,\delta}-\nabla_x\varphi^{\epsilon,\delta})(x,y,t,s)\;\mathrm{d}w}_{H_4}.
\end{aligned}$$

Here we have also used the notation

$$\hat{\mathcal{L}}[\phi](x)=\hat{\mathcal{L}}_r[\phi](x)+\hat{\mathcal{L}}^r[\phi](x)+\gamma^{\mu,r}\cdot\hat{D}_{\Delta x}\phi(x)$$

where $\hat{\mathcal{L}}$ is defined in (3.3), $\hat{\mathcal{L}}^r=\mathcal{L}^r$ for $r\geq\frac{\Delta x}{2}$, and

$$\hat{\mathcal{L}}_r[\phi](x)=\int_{\frac{\Delta x}{2}<|z|<r}\phi(x+z)-\phi(x)-\mathbf{1}_{|z|<1}z\cdot\hat{D}_{\Delta x}\phi(x)\;\mathrm{d}\mu(z).$$

Note that the discrete operator $\hat{D}_l=\hat{D}_{\Delta x,l}$ (see (3.2)) always acts on the x-variable (the variable of $\bar{u}$). To complete the proof we need to estimate $H_1,\ldots,H_4$.

2. *Estimates of H_1 and H_2.* By Taylor's formula with integral remainder, integration by parts, and Fubini (– see e.g. Lemma B.1 in [2] for more details),

$$\begin{aligned}
|H_1| &\leq \iint_{Q_T}\iint_{Q_T}\int_{|z|\leq r}\int_0^1(1-\tau)\left|D_y\eta(A(\bar{u}(x,t)),A(u(y,s)))\right|\\
&\qquad\qquad\cdot\,\omega_\delta(t-s)\,\underbrace{\left|D_y\Omega_\epsilon(x-y+\tau z)\right|}_{=|D_x\Omega_\epsilon(x-y+\tau z)|}|z|^2\;\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}w\\
&\leq\frac{1}{2}L_A\int_0^T|u(\cdot,s)|_{BV(\mathbb{R}^d)}\,\mathrm{d}s\int_{\mathbb{R}^d}|D_x\Omega_\epsilon(x)|\;\mathrm{d}x\int_{\mathbb{R}}\omega_\varepsilon(t)\,\mathrm{d}t\int_{|z|\leq r}|z|^2\;\mathrm{d}\mu(z))\\
&\leq C_T\,L_A\,|u_0|_{BV(\mathbb{R}^d)}\,\epsilon^{-1}\int_{|z|\leq r}|z|^2\,\mathrm{d}\mu(z).
\end{aligned}$$

Here we also used Theorem 2.1 and the standard estimate $\int_{\mathbb{R}^d}|D_x\Omega_\epsilon(x)|\;\mathrm{d}x=\mathcal{O}(\frac{1}{\varepsilon})$.

We find a similar estimate for H_2 via a regularization procedure and the argument for H_1 above. Let $\bar{\varphi}_\varrho^{\epsilon,\delta}$ be a mollification in the x-variable of $\bar{\varphi}^{\epsilon,\delta}$, i.e. $\bar{\varphi}_\varrho^{\epsilon,\delta}=\bar{\varphi}^{\epsilon,\delta}*_x\Omega_\rho$ where the convolution is in x only. Then $\bar{\varphi}_\varrho^{\epsilon,\delta}$ is smooth in x, and

$$|\bar{\varphi}_\varrho^{\epsilon,\delta}(\cdot,y,t,s)|_{BV(\mathbb{R}^d)}\leq|\bar{\varphi}^{\epsilon,\delta}(\cdot,y,t,s)|_{BV(\mathbb{R}^d)}\leq|\varphi^{\epsilon,\delta}(\cdot,y,t,s)|_{BV(\mathbb{R}^d)}=O\left(\epsilon^{-1}\right),$$

where the first inequality holds for all ϱ small enough (cf. e.g. [30, Theorem 5.3.1]), while the second one is obvious. Let us call

$$H_2^\varrho=\iint_{Q_T}\iint_{Q_T}\eta(A(\bar{u}(x,t)),A(u(y,s)))\,\hat{\mathcal{L}}_r^{\mu^*}[\bar{\varphi}_\varrho^{\epsilon,\delta}(\cdot,y,t,s)](x)\;\mathrm{d}w.$$

First note that $\lim_{\varrho\to 0} H_2^\varrho = H_2$ by the dominated convergence theorem since we are integrating away from the singularity and $\bar\varphi_\varrho^{\epsilon,\delta}(\cdot,y,t,s)\to\bar\varphi^{\epsilon,\delta}(\cdot,y,t,s)$ pointwise. Then, since $\bar\varphi_\varrho^{\epsilon,\delta}(\cdot,y,t,s)$ is smooth, we repeat the argument used for H_1 and obtain

$$|H_2^\varrho| \leq C_T\, L_A\, |u_0|_{BV(\mathbb{R}^d)}\, |\bar\varphi_\varrho^{\epsilon,\delta}|_{BV(\mathbb{R}^d)} \int_{\frac{\Delta x}{2}<|z|\leq r} |z|^2 \,\mathrm{d}\mu(z).$$

Since $|\bar\varphi_\varrho^{\epsilon,\delta}|_{BV(\mathbb{R}^d)} = O(\varepsilon^{-1})$, we can take the limit $\varrho\to 0$ and get

$$|H_2| \leq C_T\, L_A\, |u_0|_{BV(\mathbb{R}^d)}\, \epsilon^{-1} \int_{|z|\leq r} |z|^2 \,\mathrm{d}\mu(z).$$

3. *Estimate of* H_3. By the definition of $\bar\varphi^{\epsilon,\delta}$ and properties of mollifiers, a standard argument shows that

$$\begin{aligned}&\iint_{Q_T} \left|\bar\varphi^{\epsilon,\delta}(x,y,t,s)-\varphi^{\epsilon,\delta}(x,y,t,s)\right| \,\mathrm{d}y\,\mathrm{d}s\\ &\leq d|\Omega_\varepsilon|_{BV}\|\omega_\delta\|_{L^1}\Delta x + d\|\Omega_\varepsilon\|_{L^1}|\omega_\delta|_{BV}\Delta t \leq O\left(\frac{\Delta x}{\epsilon}+\frac{\Delta t}{\delta}\right).\end{aligned}$$

Similar estimates are given in e.g. [13]. This estimate along with several applications of Fubini's theorem then show that for all $\frac{\Delta x}{2} < r \leq 1$,

$$\begin{aligned}|H_3| &\leq \iint_{Q_T} \left|\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\right| \left(\iint_{Q_T}\left|\bar\varphi^{\epsilon,\delta}(x,y,t,s)-\varphi^{\epsilon,\delta}(x,y,t,s)\right| \,\mathrm{d}y\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t\\
&\leq c\,L_A\left(\frac{\Delta x}{\epsilon}+\frac{\Delta t}{\delta}\right)\left(\iint_{Q_T}\int_{r<|z|\leq 1}|\bar u(x+z,t)-\bar u(x,t)|\,\mathrm{d}\mu(z)\,\mathrm{d}x\,\mathrm{d}t\right.\\
&\qquad\left.+\iint_{Q_T}\int_{|z|>1}|\bar u(x+z,t)-\bar u(x,t)|\,\mathrm{d}\mu(z)\,\mathrm{d}x\,\mathrm{d}t\right)\\
&\leq C_T L_A\Big(\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\Big)\left(|u_0|_{BV}\int_{r<|z|\leq 1}|z|\,\mathrm{d}\mu(z)+\|u_0\|_{L^1}\int_{|z|>1}\mathrm{d}\mu(z)\right).\end{aligned}$$

4. *Estimate of* H_4. Let $l\in(0,\dots,d)$ and write

$$\begin{aligned}H_{4,l} &= \gamma_l^{\mu^*,r}\underbrace{\iint_{Q_T}\sum_{\alpha\in\mathbb{Z}^d}\sum_{n=0}^{N-1}\eta(A(U_\alpha^n),A(u(y,s)))\int_{t_n}^{t_{n+1}}\int_{R_\alpha}\hat D_l\bar\varphi^{\epsilon,\delta}(x,y,t,s)\,\mathrm{d}w}_{H_{4,l}^1}\\
&\quad-\gamma_l^{\mu^*,r}\underbrace{\iint_{Q_T}\sum_{\alpha\in\mathbb{Z}^d}\sum_{n=0}^{N-1}\eta(A(U_\alpha^n),A(u(y,s)))\int_{t_n}^{t_{n+1}}\int_{R_\alpha}\partial_{x_l}\varphi^{\epsilon,\delta}(x,y,t,s)\,\mathrm{d}w}_{H_{4,l}^2}.\end{aligned}$$

Since $\int_{t_n}^{t_{n+1}}\int_{R_\alpha}\bar\varphi^{\epsilon,\delta}(x,y,t,s)\,\mathrm{d}x\mathrm{d}t = \int_{t_n}^{t_{n+1}}\int_{R_\alpha}\varphi^{\epsilon,\delta}(x,y,t_{n+1},s)\,\mathrm{d}x\mathrm{d}t$ by definition, we can use summation by parts to find that

$$H_{4,l}^1 = -\iint_{Q_T}\sum_{\alpha\in\mathbb{Z}^d}\sum_{n=0}^{N-1}\hat D_l\eta(A(U_\alpha^n),A(u(y,s)))\int_{t_n}^{t_{n+1}}\int_{R_\alpha}\varphi^{\epsilon,\delta}(x,y,t_{n+1},s)\,\mathrm{d}w.$$

Integration in the x_l-direction followed by summation by parts leads to

$$H_{4,l}^2 = -\Delta x \iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d} \sum_{n=0}^{N-1} \hat{D}_l \eta(A(U_\alpha^n), A(u(y,s))) \cdot \int_{t_n}^{t_{n+1}} \int \cdots \int \varphi^{\epsilon,\delta}(x_{|x_l=x_{\alpha_l}}, y, t, s)\, \mathrm{d}x_1 \dots \mathrm{d}x_{l-1}\, \mathrm{d}x_{l+1} \dots \mathrm{d}x_d\, \mathrm{d}t\, \mathrm{d}y\, \mathrm{d}s.$$

Here we first integrated $\partial_{x_l}\varphi^{\epsilon,\delta}(\cdot, y, t, s)$ along the interval $(x_{\alpha_l}, x_{\alpha_{l+1}})$ to obtain the difference $\varphi^{\epsilon,\delta}(x_{|x_l=x_{\alpha_{l+1}}}, y, t, s) - \varphi^{\epsilon,\delta}(x_{|x_l=x_{\alpha_l}}, y, t, s)$, and then we used summation by parts to move this difference onto $\eta(A(U_\alpha^n), A(u(y,s)))$. Note that $x_{|x_l=x_{\alpha_l}} = (x_1, \dots, x_{l-1}, x_{\alpha_l}, x_{l+1}, \dots, x_d)$, and that $x_l = x_{\alpha_l}$ is fixed here while the other variables $x_j,\ j \neq l$ vary.

By the above computations, the inequality $|\hat{D}_l\eta(A(U_\alpha^n), A(u(y,s)))| \leq |\hat{D}_l A(U_\alpha^n)|$ (i.e. $||a-k| - |b-k|| \leq |a-b|$), and Fubini, we find that

$$\begin{aligned} H_{4,l} &= \gamma_l^{\mu^*,r} \iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d} \sum_{n=0}^{N-1} \hat{D}_l \eta(A(U_\alpha^n), A(u(y,s))) \\ &\quad \cdot \int_{t_n}^{t_{n+1}} \int_{R_\alpha} \Big(\varphi^{\epsilon,\delta}(x_{|x_l=x_{\alpha_l}}, y, t, s) - \varphi^{\epsilon,\delta}(x_\alpha, y, t_{n+1}, s)\Big)\, \mathrm{d}x\, \mathrm{d}t\ \mathrm{d}y\, \mathrm{d}s \\ &\leq \gamma_l^{\mu^*,r} \sum_{\alpha\in\mathbb{Z}^d} \sum_{n=0}^{N-1} |\hat{D}_l A(U_\alpha^n)| \\ &\quad \cdot \int_{t_n}^{t_{n+1}} \int_{R_\alpha} \iint_{Q_T} \Big|\varphi^{\epsilon,\delta}(x_{|x_l=x_{\alpha_l}}, y, t, s) - \varphi^{\epsilon,\delta}(x_\alpha, y, t_{n+1}, s)\Big|\, \mathrm{d}w. \end{aligned}$$

Since $\phi^{\varepsilon,\delta}(x,y,t,s) = \Omega_\varepsilon(x-y)\omega_\delta(t-s)$ and $(x,t) \in R_\alpha \times (t_n, t_{n+1}]$, we find as in part 3 that

$$\begin{aligned} &\iint_{Q_T} \Big|\varphi^{\epsilon,\delta}(x_{|x_l=x_{\alpha_l}}, y, t, s) - \varphi^{\epsilon,\delta}(x_\alpha, y, t_{n+1}, s)\Big|\ \mathrm{d}y\mathrm{d}s \\ &\leq C\Big(|\Omega_\varepsilon|_{BV}\|\omega_\delta\|_{L^1}\Delta x + \|\Omega_\varepsilon\|_{L^1}|\omega_\delta|_{BV}\Delta t\Big) = O\left(\frac{\Delta x}{\varepsilon} + \frac{\Delta t}{\delta}\right). \end{aligned}$$

Summing over l we then find that

$$|H_4| \leq d\, C |\gamma^{\mu^*,r}| \left(\frac{\Delta t}{\delta} + \frac{\Delta x}{\epsilon}\right) \Big(\sum_{n=0}^{N-1} \sum_{\alpha\in\mathbb{Z}^d} |\hat{D}_l A(U_\alpha^n)| \Delta t\, \Delta x^d\Big),$$

and since $\sum_{\alpha\in\mathbb{Z}^d} |\hat{D}_l A(U_\alpha^n)|\Delta x^d = |A(\bar{u}(\cdot, t_n))|_{BV} \leq L_A |u_0|_{BV}$, we conclude that

$$|H_4| \leq C_T L_A \left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\right) \int_{r<|z|\leq 1} |z|\, \mathrm{d}\mu(z).$$

In view of part 1 - 4 the proof is now complete. □

Proof of Theorem 6.1 for the explicit method (3.6). We argue as in the beginning of the proof for the implicit method, replacing the implicit cell entropy inequality by the explicit one (4.6), and find that

$$
\begin{aligned}
&\|u(\cdot,T)-\bar u(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C_T\left(\Delta x+\epsilon+\mathcal{E}_\delta(u)\vee\mathcal{E}_\delta(v)\right)\\
&\quad+\iint_{Q_T}\iint_{Q_T}\eta(A(\bar u(x,t)),A(u(y,s)))\,\mathcal{L}_r^{\mu^*}[\varphi^{\epsilon,\delta}(x,\cdot,t,s)](y)\;\mathrm{d}w\\
&\quad+\iint_{Q_T}\iint_{Q_T}\eta(A(\bar u(x,t)),A(u(y,s)))\,\hat{\mathcal{L}}_r^{\mu^*}[\bar\varphi^{\epsilon,\delta}(\cdot,y,t,s)](x)\;\mathrm{d}w\\
&\quad+\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t+\Delta t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&\quad-\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&\quad+\iint_{Q_T}\iint_{Q_T}\eta(A(\bar u(x,t)),A(u(y,s)))\,\gamma^{\mu^*,r}\cdot(\hat D\bar\varphi^{\epsilon,\delta}-\nabla_x\varphi^{\epsilon,\delta})(x,y,t,s)\;\mathrm{d}w.
\end{aligned}
$$

The difference with the previous proof is the interpolation (5.2), and more importantly, the new $\mathcal{L}^{\mu,r}$-terms. Note that by a change of variables,

$$
\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t+\Delta t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&=\iint_{Q_T}\iint_{Q_{\Delta t,T}}\eta'(\bar u(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t-\Delta t))](x)\,\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&\quad+\int_T^{T+\Delta t}\int_{\mathbb{R}^d}\iint_{Q_T}\eta'(\bar u(x,t),u(y,s))\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t-\Delta t))](x)\,\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w,
\end{aligned}
$$

where $Q_{a,b}=\mathbb{R}^d\times(a,b)$. The last term on the right can be estimated by

$$
\begin{aligned}
&\Delta t\|\phi^{\varepsilon,\delta}\|_{L^1}\left(|A(\bar u)|_{BV}\int_{r<|z|<1}|z|\,\mathrm{d}\mu(z)+2\|A(\bar u)\|_{L^1}\int_{|z|>1}\mathrm{d}\mu(z)\right)\\
&=O(\Delta t)\int_{r<|z|<1}|z|\,\mathrm{d}\mu(z).
\end{aligned}
$$

By similar computations, we can write the $\mathcal{L}^{\mu,r}$-terms in the above inequality as

$$
\begin{aligned}
&\underbrace{\iint_{Q_T}\iint_{Q_{\Delta t,T}}\eta'(\bar u(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t-\Delta t))-A(\bar u(\cdot,t))](x)\,\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w}_{I}\\
&+\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,(\bar\varphi^{\epsilon,\delta}-\varphi^{\epsilon,\delta})(x,y,t,s)\;\mathrm{d}w\\
&+O(\Delta t)\int_{r<|z|<1}|z|\,\mathrm{d}\mu(z).
\end{aligned}
$$

Here we estimate the first term using the time regularity of $\bar u$,

$$
\begin{aligned}
I&\le\iint_{Q_T}|\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t-\Delta t))-A(\bar u(\cdot,t))](x)|\underbrace{\iint_{Q_T}\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w}_{=O(1)}\\
&\le c\,2\,L_A\left(\int_0^T\|\bar u(\cdot,t-\Delta t)-\bar u(\cdot,t)\|_{L^1(\mathbb{R}^d)}\;\mathrm{d}t\right)\int_{|z|>r}\mathrm{d}\mu(z)\\
&\le C_T\,\mathcal{E}_{\Delta t}(\bar u)\int_{|z|>r}\mathrm{d}\mu(z),
\end{aligned}
$$

where $\mathcal{E}_{\Delta t}(\bar u)$ is defined in (2.2). Now all the remaining terms can be estimated as in the proof for the implicit method (3.5), so the proof is complete. □

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